Investigating the representation and congruence extension properties for quantales

Background in the theory of partially ordered monoids

Definition 1 By a partially ordered monoid (briefly, pomonoid) we mean a monoid S endowed with a partial order $\leq$ such that

$$\forall s_1, s_2, t_1, t_2 \in S,\ s_1 \leq s_2,\ t_1 \leq t_2 \Longrightarrow s_1 s_2 \leq t_1 t_2.$$

We then say that $\leq$ is compatible with the binary operation of S.

Let X be a poset. Then the set $\mathcal{O}(X)$ of all monotone transformations of X is a pomonoid with respect to the usual compositions of transformations and pointwise order. Given a pomonoid S and a poset X, a (pomonoid) homomorphism $\gamma : S \longrightarrow \mathcal{O}(X)$ is called an ordered representation of S by $\mathcal{O}(X)$. Let us introduce the notation $\gamma_s = (s)\gamma$, $s \in S$. Then, it can be easily verified that

$$x \cdot s = (x)\,\gamma_s$$

defines a right S-action on X that is monotone in both the variables. So, every ordered representation of S gives rise to a right (equivalently, left) S-poset. The converse is also true: every right (equivalently, left) S-poset gives an ordered representation of S.

Definition 2 Let U be a subpomonoid of a pomonoid S. Then U is said to have the ordered representation extension property (POREP) in S if for every ordered representation $\gamma : U \longrightarrow \mathcal{O}(X)$, given by $u \longmapsto \gamma_u$, there exists an ordered representation $\alpha : S \longrightarrow \mathcal{O}(Y)$, given by $s \longmapsto \alpha_s$, such that

1. X is a subposet of Y, and
2. for all $u \in U$, we have $\alpha_u \mid_X = \gamma_u$.

Proposition 3 *A subpomonoid U of a pomonoid S has POREP in S, if and only if for every right U-poset X_U the canonical mapping $X \longrightarrow X \otimes_U S$ (viz. $x \longmapsto x \otimes 1$) is an order-embedding (of posets).*

Let S be a pomonoid. By an *order-congruence* on S we mean a congruence σ on the underlaying monoid for which the quotient monoid S/σ can be equipped with a partial order such that the canonical homomorphism $\sigma^\natural : S \longrightarrow S/\sigma$ is monotone.

Definition 4 Let U be a subpomonoid of a pomonoid S. Then U is said to have the right order-congruence extension property (PORCEP) in S if for every right order-congruence θ on U and every compatible partial order $\leq_\theta$ on U/θ there exists a right order-congruence Θ on S together with a compatible order $\leq_\Theta$ on S such that for all $u_1, u_2 \in U$

$$[u_1]_\theta \leq_\theta [u_1]_\theta \quad \text{iff} \quad [u_1]_\Theta \leq_\Theta [u_1]_\Theta\,.$$

Theorem 5 *If a subpomonoid U has (POREP) in a pomonoid S then it has (PORCEP) in S.*

Definition 6 A pomonoid amalgam is a triple $\mathcal{A} \equiv (U; S_1, S_2)$ of pomonoids such that $U = S_1 \cap S_2$. We say that $\mathcal{A}$ is embeddable if there exists a pomonoid W embedding S_1 and S_2 so that their intersection in W 'coincides' with U.

Definition 7 Let U be a subpomonoid of a pomonoid S. Then the pair $(U; S)$ is called an amalgamation pair if for every pomonoid T, the amalgam $(U; S, T)$ is embeddable.

Theorem 8 *If $(U; S)$ is an amalgamation pair then U has POREP in S.*

Definition 9 A (unital) quantale Q is a pomonoid that is also a complete sup-lattice such that for all $S \subseteq Q$

$$a\,(\vee S) = \ \vee\,(aS)\,,$$

and

$$(\vee S)\,b = \ \vee\,(Sb)\,.$$

Definition 10 Let Q be a quantale. By a right Q-module we mean a sup-lattice L equipped with a right Q-action, $(a, x) \longmapsto ax$, $a \in Q$, $x \in L$, such that,

1. $ab(x) = a(b(x))$.
2. $1x = x$.

Aims and scope

The aim of this research is to find the analogues of POREP, PORCEP in the context of quantales and explore the relationships among these properties and amalgamation.

References

1. Pedro Resende. Lectures on étale groupoids, inverse semigroups and quantales. Technical report Instituto Superior Técnico, University of Lisbon (2006)

2. Nasir Sohail. On amalgamation of partially ordered monoids. PhD Thesis. Submitted to Quaid-i-Azam University, Islamabad, Pakistan (2010)