Journal of Functional Analysis 280 (2021) 108846

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

On Daugavet indices of thickness ☆

Rainis Haller [a], Johann Langemets [a], Vegard Lima [b], Rihhard Nadel [a], Abraham Rueda Zoca [c,*]

[a] *Institute of Mathematics and Statistics, University of Tartu, Narva mnt 18, 51009, Tartu, Estonia*
[b] *Department of Engineering Sciences, University of Agder, Postboks 509, 4898 Grimstad, Norway*
[c] *Universidad de Murcia, Departamento de Matemáticas, Campus de Espinardo 30100 Murcia, Spain*

A R T I C L E I N F O

Article history:
Received 15 May 2020
Accepted 25 October 2020
Available online 11 November 2020
Communicated by T. Schlumprecht

MSC:
primary 46B20, 46B22

Keywords:
Daugavet property
Thickness
Diameter 2 property
Slice

A B S T R A C T

Inspired by R. Whitley's thickness index the last named author recently introduced the Daugavet index of thickness of Banach spaces. We continue the investigation of the behavior of this index and also consider two new versions of the Daugavet index of thickness, which helps us solve an open problem which connects the Daugavet indices with the Daugavet equation. Moreover, we will improve formerly known estimates of the behavior of Daugavet index on direct sums of Banach spaces by establishing sharp bounds. As a consequence of our results we prove that, for every $0 < \delta < 2$, there exists a Banach space where the infimum of the diameter of convex combinations of slices of the unit ball is exactly δ, solving an open question from the literature. Finally, we prove that an open question posed by Ivakhno in 2006 about

☆ This work was supported by the Estonian Research Council grants (PRG877) and (PSG487). R. Nadel was supported in part by the University of Tartu Foundation's CWT Estonia travel scholarship. A. Rueda Zoca is supported by MICINN (Spain) Grant PGC2018-093794-B-I00 (MCIU, AEI, FEDER, UE), by Junta de Andalucía Grant A-FQM-484-UGR18 and by Junta de Andalucía Grant FQM-0185.

* Corresponding author.
E-mail addresses: rainis.haller@ut.ee (R. Haller), johann.langemets@ut.ee (J. Langemets), Vegard.Lima@uia.no (V. Lima), rihhard.nadel@ut.ee (R. Nadel), abrahamrueda@ugr.es (A. Rueda Zoca).
URLs: https://johannlangemets.wordpress.com/ (J. Langemets), https://arzenglish.wordpress.com (A. Rueda Zoca).

https://doi.org/10.1016/j.jfa.2020.108846

the relation between the radius and diameter of slices has a negative answer.

1. Introduction

Let X be a real Banach space. To measure quantitatively how far X is from having the Daugavet property, the last named author introduced in [13] a parameter $\mathcal{T}(X)$, called the Daugavet index of thickness of X (for the regular index of thickness see [17]), where

$$\mathcal{T}(X) = \inf \left\{ r > 0 \;\middle|\; \begin{array}{l} \text{there exist } x \in S_X \text{ and a relatively weakly} \\ \text{open } W \text{ in } B_X \text{ such that } \emptyset \neq W \subset B(x,r) \end{array} \right\}.$$

Notice that $0 \leq \mathcal{T}(X) \leq 2$ for any Banach space X and, for example, $\mathcal{T}(\ell_1) = 0$, $\mathcal{T}(c_0) = \mathcal{T}(\ell_\infty) = 1$, and $\mathcal{T}(C[0,1]) = 2$ [13, Example 4.3]. In fact, $\mathcal{T}(X) = 2$ holds if and only if X has the Daugavet property [15, Lemmata 2 and 3]. Clearly, $\mathcal{T}(X) = 0$ for X with the Radon–Nikodým property. The converse does not hold in general, because there exists a Banach space X, where every slice of B_X has diameter two (and therefore X does not have the Radon–Nikodým property), but with arbitrarily small nonempty relatively weakly open subsets of B_X (hence $\mathcal{T}(X) = 0$) [2, Theorem 2.4].

Clearly, a slice of the unit ball is relatively weakly open. On the other hand, by Bourgain's lemma [7, Lemma II.1], every nonempty relatively weakly open subset of the unit ball contains a convex combination of slices. Moreover, there exists a Banach space such that every nonempty relatively weakly open subset of the unit ball has diameter two, but it also contains convex combination of slices with arbitrarily small diameter [4, Theorem 2.5].

The previous two examples motivate us to study further the index $\mathcal{T}(\cdot)$ and to introduce two new related Daugavet indices, which are in general not equal:

$$\mathcal{T}^s(X) = \inf \left\{ r > 0 \;\middle|\; \begin{array}{l} \text{there exist } x \in S_X \text{ and a slice } S \text{ of } B_X \\ \text{such that } S \subset B(x,r) \end{array} \right\}$$

and

$$\mathcal{T}^{cc}(X) = \inf \left\{ r > 0 \;\middle|\; \begin{array}{l} \text{there exist } x \in S_X \text{ and a convex} \\ \text{combination } C \text{ of relatively weakly open} \\ \text{subsets of } B_X \text{ such that } \emptyset \neq C \subset B(x,r) \end{array} \right\}.$$

Remark 1.1. If one replaces relatively weakly open subsets of B_X in the definition of $\mathcal{T}^{cc}(X)$ with slices of B_X, then, by Bourgain's lemma, the index remains unchanged.

Observe that

$$0 \leq \mathcal{T}^{cc}(X) \leq \mathcal{T}(X) \leq \mathcal{T}^{s}(X) \leq 2.$$

Moreover, if in a Banach space X every slice (respectively, nonempty relatively weakly open subset; convex combination of nonempty relatively weakly open subsets) of B_X has diameter two, then $\mathcal{T}^s(X) \geq 1$ (respectively, $\mathcal{T}(X) \geq 1$; $\mathcal{T}^{cc}(X) \geq 1$).

It is known that $\mathcal{T}^{cc}(X) = \mathcal{T}(X) = \mathcal{T}^s(X) = 2$ if and only if X has the Daugavet property. This is immediate from the following result.

Proposition 1.2 *(see [15, Lemmata 2 and 3]). Let X be a Banach space. The following assertions are equivalent:*

(i) X has the Daugavet property;
(ii) For every $x \in S_X$, every $\varepsilon > 0$, and every slice S of B_X there exists $y \in S$ such that $\|x - y\| \geq 2 - \varepsilon$;
(iii) For every $x \in S_X$, every $\varepsilon > 0$, and every nonempty relatively weakly open subset W of B_X there exists $y \in W$ such that $\|x - y\| \geq 2 - \varepsilon$;
(iv) For every $\varepsilon > 0$, every $x \in S_X$ and every convex combination C of nonempty relatively weakly open subsets of B_X, there exists $y \in C$ such that $\|x - y\| > 2 - \varepsilon$.

The statement (iv) in the above proposition is not explicit in [15], but as pointed out in [5, Example 3.3] the equivalence of (i) and (iv) is clear from the proof of [15, Lemma 3].

Examples of Banach spaces with the Daugavet property include $\mathcal{C}(K, X)$ (resp. $L_1(\mu, X)$ and $L_\infty(\mu, X)$), regardless X, where K has no isolated points and μ contains no atom [16], the ℓ_1-sum and the ℓ_∞-sum of two Banach spaces with the Daugavet property [18] or $C[0,1]\widehat{\otimes}_\pi C[0,1]$ [14, Theorem 1.2].

Of course, in general one has

$$\mathcal{T}^s(X) \leq \inf \left\{ r > 0 \;\middle|\; \begin{array}{l} \text{there exist a slice } S \text{ of } B_X \text{ and } x \in S \cap S_X \\ \text{such that } S \subset B(x, r) \end{array} \right\}$$

and

$$\mathcal{T}(X) \leq \inf \left\{ r > 0 \;\middle|\; \begin{array}{l} \text{there exist a relatively weakly open } W \text{ in } B_X \\ \text{and } x \in W \cap S_X \text{ such that } W \subset B(x, r) \end{array} \right\}.$$

However, notice that both of these inequalities can be strict. For example, if $X = C[0,1] \oplus_2 C[0,1]$, then both right hand side inequalities are 2, but $\mathcal{T}(X) \leq \mathcal{T}^s(X) < 2$. This happens for any Banach space X, which fails the Daugavet property, but has the diametral diameter two property (see the definition in [5]).

In Section 2 we carry out a systematic study of Daugavet indices of thickness in direct sums of Banach spaces. We establish sharp bounds on all of the indices in ℓ_p-sums (see Theorem 2.6), which improve the known upper estimates from [13]. As an application, we prove that for each $r \in [0,2]$ there exists a Banach space X such that $\mathcal{T}^s(X) = \mathcal{T}(X) = \mathcal{T}^{cc}(X) = r$ (see Theorem 2.7). For the proof, we make use of Proposition 2.8, which allows us to solve in Corollary 2.9 an open question from [10] (see the Remark after Theorem 2.8).

In Section 3 we answer negatively a question posed by the last named author in [13, Problem 5.3]. Also, we will discuss the relation of $\mathcal{T}^s(\cdot)$ between isomorphic Banach spaces (see Proposition 3.4), which is then applied to prove that the Daugavet property is closed with respect to the Banach–Mazur distance.

We end the paper by giving a negative answer to a question of Ivakhno from 2006 [11, p. 96]. If in a Banach space X every slice of B_X has diameter two, then every slice has radius one, that is, X has the r-big slice property (see Section 3 for details). Ivakhno asked whether the converse is always true. We show in Theorem 3.7 that there exists a Banach space X such that X has the r-big slice property, but for every $\varepsilon > 0$ the unit ball contains slices of diameter less than $\sqrt{2} + \varepsilon$, which answers the above mentioned question negatively.

All Banach spaces considered in this paper are nontrivial and over the real field. The closed unit ball of a Banach space X is denoted by B_X and its unit sphere by S_X. The dual space of X is denoted by X^* and the bidual by X^{**}.

By a *slice* of B_X we mean a set of the form

$$S(B_X, x^*, \alpha) := \{x \in B_X : x^*(x) > 1 - \alpha\},$$

where $x^* \in S_{X^*}$ and $\alpha > 0$. A finite convex combination of slices is then of the form

$$\sum_{i=1}^{n} \lambda_i S(B_X, x_i^*, \alpha_i),$$

where $n \in \mathbb{N}$ and $\lambda_i \in [0,1]$ such that $\sum_{i=1}^{n} \lambda_i = 1$.

We recall that a norm N on $\mathbb{R}^2$ is called *absolute* (see [1]) if

$$N(a,b) = N(|a|,|b|) \qquad \text{for all } (a,b) \in \mathbb{R}^2$$

and *normalized* if

$$N(1,0) = N(0,1) = 1.$$

For example, the ℓ_p-norm $\|\cdot\|_p$ is absolute and normalized for every $p \in [1,\infty]$. If N is an absolute normalized norm on $\mathbb{R}^2$ (see [1, Lemmata 21.1 and 21.2]), then

- $\|(a,b)\|_\infty \le N(a,b) \le \|(a,b)\|_1$ for all $(a,b) \in \mathbb{R}^2$;

- if $(a,b),(c,d) \in \mathbb{R}^2$ with $|a| \leq |c| \quad \text{and} \quad |b| \leq |d|$, then

$$N(a,b) \leq N(c,d);$$

- the dual norm N^* on $\mathbb{R}^2$ defined by

$$N^*(c,d) = \max_{N(a,b)\leq 1}(|ac| + |bd|) \qquad \text{for all } (c,d) \in \mathbb{R}^2$$

  is also absolute and normalized. Note that $(N^*)^* = N$.

If X and Y are Banach spaces and N is an absolute normalized norm on $\mathbb{R}^2$, then we denote by $X \oplus_N Y$ the product space $X \times Y$ with respect to the norm

$$\|(x,y)\|_N = N(\|x\|, \|y\|) \qquad \text{for all } x \in X \text{ and } y \in Y.$$

In the special case where N is the ℓ_p-norm, we write $X \oplus_p Y$. Note that $(X \oplus_N Y)^* = X^* \oplus_{N^*} Y^*$.

2. Daugavet indices in direct sums

We start by recalling a result for the index $\mathcal{T}(\cdot)$ from [13].

Proposition 2.1 *(see [13, Proposition 4.5]). Let X and Y be Banach spaces. Then*

(a) $\mathcal{T}(X \oplus_1 Y) \leq \min\{\mathcal{T}(X), \mathcal{T}(Y)\}$;
(b) $\mathcal{T}(X \oplus_p Y) \leq (\frac{(2^{1/p}+1)^p+1}{2})^{1/p}$ *for every* $1 < p < \infty$;
(c) $\mathcal{T}(X \oplus_\infty Y) \geq \min\{\mathcal{T}(X), \mathcal{T}(Y)\}$, *where equality holds if* $\mathcal{T}(X \oplus_\infty Y) > 1$.

Now we provide a lower estimate for $\mathcal{T}(X \oplus_p Y)$, where $1 < p < \infty$.

Proposition 2.2. *Let X and Y be Banach spaces, N be an absolute normalized norm on $\mathbb{R}^2$, and $\gamma > 0$ be such that $N(\cdot) \geq \gamma\|\cdot\|_1$. Then*

(a) $\mathcal{T}^s(X \oplus_N Y) \geq 2\gamma\big(\min\{\mathcal{T}^s(X), \mathcal{T}^s(Y)\} - 1\big)$;
(b) $\mathcal{T}(X \oplus_N Y) \geq 2\gamma\big(\min\{\mathcal{T}(X), \mathcal{T}(Y)\} - 1\big)$;
(c) $\mathcal{T}^{cc}(X \oplus_N Y) \geq 2\gamma\big(\min\{\mathcal{T}^{cc}(X), \mathcal{T}^{cc}(Y)\} - 1\big)$.

In particular, $\mathcal{T}(X \oplus_p Y) \geq 2^{1/p}\big(\min\{\mathcal{T}(X), \mathcal{T}(Y)\} - 1\big)$ *whenever* $1 < p < \infty$.

Proof. We will only prove (b), because the proofs of (a) and (c) are very similar.

(b). Without loss of generality we assume that $\min\{\mathcal{T}(X), \mathcal{T}(Y)\} > 1$ otherwise the lower bound trivially holds. Let $\varepsilon > 0$ be such that $\min\{\mathcal{T}(X), \mathcal{T}(Y)\} > 1 + \varepsilon$. Denote

$Z := X \oplus_N Y$. Let $(\tilde{x}, \tilde{y}) \in S_Z$ and let W be a nonempty relatively weakly open subset of B_Z.

Without loss of generality we may assume that

$$W = \{z \in B_Z \colon |z_i^*(z) - z_i^*(z_0)| < 1,\ i \in \{1, \dots, n\}\},$$

for some $z_i^* = (x_i^*, y_i^*) \in Z^*$ and $z_0 = (x_0, y_0) \in S_Z$.

Define now

$$U := \begin{cases} \{x \in B_X \colon |x_i^*(x) - x_i^*(\frac{x_0}{\|x_0\|})| < \frac{1}{2\|x_0\|},\ i \in \{1, \dots, n\}\}, & \text{if } x_0 \neq 0, \\ \{x \in B_X \colon |x_i^*(x)| < \frac{1}{2},\ i \in \{1, \dots, n\}\}, & \text{if } x_0 = 0, \end{cases}$$

and

$$V := \begin{cases} \{y \in B_Y \colon |y_i^*(y) - y_i^*(\frac{y_0}{\|y_0\|})| < \frac{1}{2\|y_0\|},\ i \in \{1, \dots, n\}\}, & \text{if } y_0 \neq 0, \\ \{y \in B_Y \colon |y_i^*(y)| < \frac{1}{2},\ i \in \{1, \dots, n\}\}, & \text{if } y_0 = 0. \end{cases}$$

From now on we will distinguish two cases.

Case 1: Assume first that $\tilde{x} \neq 0$ and $\tilde{y} \neq 0$. Due to the definition of the Daugavet index we can find $u \in U$ and $v \in V$ such that $\|\frac{\tilde{x}}{\|\tilde{x}\|} - u\| \geq \mathcal{T}(X) - \varepsilon/2$ and $\|\frac{\tilde{y}}{\|\tilde{y}\|} - v\| \geq \mathcal{T}(Y) - \varepsilon/2$.

Claim. If two elements e and $\tilde{e}$ of the unit ball of a Banach space E satisfy that $\|e + \tilde{e}\| \geq 1 + \alpha$ for some $\alpha \in [0, 1]$, then $\|\lambda e + \mu \tilde{e}\| \geq (\lambda + \mu)\alpha$ for all $\lambda, \mu \geq 0$. Indeed, the claim follows from the inequalities

$$(\lambda + \mu)(1 + \alpha) \leq (\lambda + \mu)\|e + \tilde{e}\| \leq \|\lambda e + \mu \tilde{e}\| + \lambda + \mu.$$

Since

$$\mathcal{T}(X) - \varepsilon/2 > 1 + \varepsilon/2,$$

we can apply the Claim above and get that

$$\|\tilde{x} - \|x_0\| u\| \geq (\|\tilde{x}\| + \|x_0\|)(\mathcal{T}(X) - 1 - \varepsilon/2).$$

Similarly, one obtains $\|\tilde{y} - \|y_0\| v\| \geq (\|\tilde{y}\| + \|y_0\|)(\mathcal{T}(Y) - 1 - \varepsilon/2)$.

Observe that $(\|x_0\| u, \|y_0\| v) \in W$ and

$$\|(\tilde{x}, \tilde{y}) - (\|x_0\| u, \|y_0\| v)\|_N = N\Big(\|\tilde{x} - \|x_0\| u\|, \|\tilde{y} - \|y_0\| v\|\Big)$$

$$
\begin{aligned}
&\geq N\Big((\|\tilde{x}\|+\|x_0\|)(\mathcal{T}(X)-1-\varepsilon/2),\\
&(\|\tilde{y}\|+\|y_0\|)(\mathcal{T}(Y)-1-\varepsilon/2)\Big)\\
&\geq N\Big(\|\tilde{x}\|+\|x_0\|,\|\tilde{y}\|+\|y_0\|\Big)(\min\{\mathcal{T}(X),\mathcal{T}(Y)\}-1-\varepsilon/2)\\
&\geq \gamma(\|\tilde{x}\|+\|\tilde{y}\|+\|x_0\|+\|y_0\|)(\min\{\mathcal{T}(X),\mathcal{T}(Y)\}-1-\varepsilon/2)\\
&\geq 2\gamma(\min\{\mathcal{T}(X),\mathcal{T}(Y)\}-1-\varepsilon/2).
\end{aligned}
$$

Hence we conclude that $\mathcal{T}(X \oplus_N Y) \geq 2\gamma(\min\{\mathcal{T}(X),\mathcal{T}(Y)\}-1)$ by the arbitrariness of ε.

Case 2: Assume now that $\tilde{y}=0$, hence $\|\tilde{x}\|=1$ (the case where $\tilde{x}=0$ can be considered similarly). Now we can find $u \in U$ such that $\|\tilde{x}-u\| \geq \mathcal{T}(X)-\varepsilon/2$ and let $v \in V \cap S_Y$. Again, note that $(\|x_0\|u, \|y_0\|v) \in W$ and

$$
\begin{aligned}
\|(\tilde{x},0)-(\|x_0\|u,\|y_0\|v)\|_N &= N\Big(\|\tilde{x}-\|x_0\|u\|,\|y_0\|\Big)\\
&\geq N\Big((1+\|x_0\|)(\mathcal{T}(X)-1-\varepsilon/2),\|y_0\|\Big)\\
&\geq N\Big(1+\|x_0\|,\frac{\|y_0\|}{(\mathcal{T}(X)-1-\varepsilon/2)}\Big)(\mathcal{T}(X)-1-\varepsilon/2)\\
&\geq \gamma(1+\|x_0\|+\|y_0\|)(\mathcal{T}(X)-1-\varepsilon/2)\\
&\geq 2\gamma(\mathcal{T}(X)-1-\varepsilon/2).
\end{aligned}
$$

Again we conclude that $\mathcal{T}(X \oplus_N Y) \geq 2\gamma(\min\{\mathcal{T}(X),\mathcal{T}(Y)\}-1)$ by the arbitrariness of ε. $\square$

Remark 2.3. With almost identical proof one can generalize Proposition 2.2 to a finite direct sum of Banach spaces equipped with an absolute norm and the estimates will remain the same.

We now turn our attention to the upper estimate of these Daugavet indices of thickness.

Proposition 2.4. *Let X and Y be Banach spaces, N be an absolute normalized norm on $\mathbb{R}^2$, and $\Gamma > 0$ is such that $N(\cdot) \leq \Gamma\|\cdot\|_\infty$. If $(1,0)$ or $(0,1)$ is an extreme point of $B_{(\mathbb{R}^2,N)}$, then $\mathcal{T}^s(X \oplus_N Y) \leq \Gamma$. In particular, $\mathcal{T}^s(X \oplus_p Y) \leq 2^{1/p}$ whenever $1 < p < \infty$.*

Proof. Denote $Z := X \oplus_N Y$ and let $\varepsilon > 0$, $x \in S_X$ and $y \in S_Y$. Assume that $e = (0,1)$ is an extreme point of $B_{(\mathbb{R}^2,N)}$ (the proof for the other case is similar). Then e is actually a strongly exposed point which allows us to fix a $\delta > 0$ such that, whenever $(a,b) \in B_{(\mathbb{R}^2,N)}$ and $b > 1-\delta$, then $|a| < \varepsilon$. Find $y^* \in S_{Y^*}$ with $y^*(y) = 1$. If $(u,v) \in S(B_Z,(0,y^*),\delta)$, then $\|v\| \geq y^*(v) > 1-\delta$. By our assumption $\|u\| < \varepsilon$. Therefore

$$\begin{aligned}\|(u,v)-(x,0)\|_N &= N\Big(\|u-x\|, \|v\|\Big)\\ &\le N\Big(1+\|u\|, \|v\|\Big)\\ &\le (1+\|u\|)N(1,1)\\ &\le \Gamma(1+\varepsilon).\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary we get $\mathcal{T}^s(Z) \le \Gamma$. □

Remark 2.5. One can also generalize Proposition 2.4 to a finite direct sum of Banach spaces equipped with an absolute norm and the estimate will remain the same.

Since $\mathcal{T}(\cdot) \le \mathcal{T}^s(\cdot)$, the obtained upper bound from Proposition 2.4 improves the previously known estimate from Proposition 2.1 (b) in [13]. Moreover, from Proposition 2.2 we know that this estimate is sharp. We summarize this in the following result.

Theorem 2.6. *Let X and Y be Banach spaces and $1 < p < \infty$. If X and Y both have the Daugavet property, then $\mathcal{T}^s(X \oplus_p Y) = \mathcal{T}(X \oplus_p Y) = \mathcal{T}^{cc}(X \oplus_p Y) = 2^{1/p}$.*

Proof. Since X and Y have the Daugavet property, $\mathcal{T}^{cc}(X) = \mathcal{T}^{cc}(Y) = 2$ by Proposition 1.2. From Proposition 2.2 we get that $\mathcal{T}^{cc}(X \oplus_p Y) \ge 2^{1/p}$ and by Proposition 2.4 we have that $\mathcal{T}^s(X \oplus_p Y) \le 2^{1/p}$, therefore

$$2^{1/p} \le \mathcal{T}^{cc}(X \oplus_p Y) \le \mathcal{T}(X \oplus_p Y) \le \mathcal{T}^s(X \oplus_p Y) \le 2^{1/p},$$

which completes the proof. □

Using Theorem 2.6 we obtain the following result.

Theorem 2.7. *For every $r \in [0,2]$ there exists a Banach space X such that $\mathcal{T}^s(X) = \mathcal{T}(X) = \mathcal{T}^{cc}(X) = r$.*

For the proof we will need the following result, but first a bit of notation. If A is a set in X^*, then A' denotes the set of all w*-limit points of A:

$$A' = \{x^* \in X^* : x^* \in \overline{A \setminus \{x^*\}}^{w^*}\}.$$

Proposition 2.8. *For every $r > 0$ there exists a Banach space X such that*

$$\mathcal{T}^{cc}(X) = \mathcal{T}(X) = \mathcal{T}^s(X) = \frac{r}{1+r}.$$

Proof. The proof is inspired by the example exhibited in [12, Theorem 2.1]. Pick an arbitrary $r > 0$. Define

$$U^* := \mathrm{co}\left(B_{\ell_1\oplus_\infty\mathbb{R}} \cup \{(0,1+r), (0,-1-r)\}\right),$$

which is clearly a weak* compact set in $(c_0 \oplus_1 \mathbb{R})^*$. Consequently, there is a norm $|||\cdot|||$ on $c_0 \oplus \mathbb{R}$ whose unit ball is

$$U := \{(x,\beta) \in c_0 \oplus \mathbb{R} : \phi(x,\beta) \le 1 \ \text{ for all } \phi \in U^*\}.$$

Consider $X := (c_0 \oplus \mathbb{R}, |||\cdot|||)$, and let us prove that X satisfies the desired requirements. It is clear that $U^* = B_{X^*}$. Also, it is clear that

$$\mathrm{ext}(U^*) = \{(0, \pm(1+r))\} \cup \{(\xi e_n, \psi 1) : n \in \mathbb{N} \text{ and } \xi, \psi \in \{-1,1\}\}. \tag{2.1}$$

Therefore, $\mathrm{ext}(U^*)' = \{(0,\pm 1)\} \subseteq \frac{1}{1+r}U^*$.

Note also that $B_{\ell_1\oplus_\infty\mathbb{R}} \subseteq U^* \subseteq (1+r)B_{\ell_1\oplus_\infty\mathbb{R}}$, so

$$\frac{1}{1+r}B_{c_0\oplus_1\mathbb{R}} \subseteq U \subseteq B_{c_0\oplus_1\mathbb{R}}.$$

Consequently, for each pair $(x,\beta) \in X$, it follows that

$$\|(x,\beta)\|_1 \le |||(x,\beta)||| \le (1+r)\|(x,\beta)\|_1. \tag{2.2}$$

Define $S_\delta := S(B_X, (0,1+r), \delta)$. Pick an element $(x,\beta) \in S_\delta$, and let us estimate $|||(x,\beta) - (0, \frac{1}{1+r})|||$. To this end, notice that

$$1 \ge (1+r)\beta = (0,1+r)(x,\beta) > 1-\delta. \tag{2.3}$$

We claim that $\|x\|_\infty \le \frac{r+\delta}{1+r}$. Assume for contradiction that there exists $n \in \mathbb{N}$ such that $|x(n)| > \frac{r+\delta}{1+r}$, choose $\xi := \mathrm{sign}(x(n))$ and define $x^* := (\xi e_n, 1) \in U^* \subseteq B_{X^*}$. Then, by (2.3),

$$1 \ge x^*((x,\beta)) = |x(n)| + \beta > \frac{r+\delta}{1+r} + \frac{1-\delta}{1+r} = \frac{r+\delta+1-\delta}{1+r} = 1,$$

a contradiction. So $\|x\|_\infty \le \frac{r+\delta}{1+r}$. Also, notice that $(0, \frac{1}{1+r}) \in S_\delta$ (note that $|||(0, \frac{1}{1+r})||| \le (1+r)\|(0, \frac{1}{1+r})\|_1 = 1$ by (2.2)).

Let us estimate $|||(x,\beta) - (0, \frac{1}{1+r})|||$. By the Krein–Milman theorem

$$|||(x,\beta) - (0, \frac{1}{1+r})||| = \sup_{x^* \in \mathrm{ext}(U^*)} |x^*((x,\beta) - (0, \frac{1}{1+r}))|.$$

Given $x^* \in \mathrm{ext}(U^*)$, then $x^* = (y^*, \lambda)$ for $y^* \in \ell_1$ and $\lambda \in \mathbb{R}$. From now on we will distinguish two cases.

Case 1: Assume first that $y^* = 0$. This implies, according to (2.1), that $|\lambda| = 1+r$. By (2.3), we get that

$$|x^*((x,\beta)-(0,\frac{1}{1+r}))| = |\lambda||\beta-\frac{1}{1+r}| \leq (1+r)\frac{\delta}{1+r} = \delta.$$

Case 2: Assume now that $y^* \neq 0$, then, by (2.1), $|\lambda| = 1$ and $y^* = \pm e_k$ for suitable $k \in \mathbb{N}$. Hence

$$\begin{aligned} |x^*((x,\beta)-(0,\frac{1}{1+r}))| &= |y^*(x)+\lambda(\beta-\frac{1}{1+r})| \\ &\leq \|x\|_\infty + |\lambda||\beta-\frac{1}{1+r}| \\ &\leq \frac{r+\delta}{1+r} + \frac{\delta}{1+r}. \end{aligned}$$

Taking into account the above cases and the fact that $\delta \leq \frac{r+\delta}{1+r} + \frac{\delta}{1+r}$ we get that

$$|||(x,\beta)-(0,\frac{1}{1+r})||| \leq \frac{r+\delta}{1+r} + \frac{\delta}{1+r}.$$

This means that $S_\delta \subseteq B((0,\frac{1}{1+r}), \frac{r+\delta}{1+r}+\frac{\delta}{1+r})$, so $\mathcal{T}^s(X) \leq \frac{r+\delta}{1+r}+\frac{\delta}{1+r}$. Since $\delta > 0$ was arbitrary we get that $\mathcal{T}^s(X) \leq \frac{r}{1+r}$.

For the second part of the proof, let us prove that $\mathcal{T}^{cc}(X) \geq \frac{r}{1+r}$, for which we will prove that every convex combination of slices of B_X has diameter at least $\frac{2r}{1+r}$. The proof will be motivated by [12, Proposition 2.2]. Take a convex combination of slices $C := \sum_{i=1}^n \lambda_i S(B_X, x_i^*, \alpha_i)$, a point $\sum_{i=1}^n \lambda_i x_i \in C$, where $x_i \in S(B_X, x_i^*, \alpha_i) \cap S_X$, and $0 < \varepsilon < \frac{r}{1+r}$. Given $i \in \{1,\ldots,n\}$ define

$$A_i := \{f \in \operatorname{ext}(B_{X^*}) : |f(x_i)| > \frac{1}{1+r} + \varepsilon\}.$$

Since $\operatorname{ext}(B_{X^*})' \subseteq \frac{1}{1+r}B_{X^*}$, a compactness argument implies that A_i is finite. Also, each A_i is non-empty since $\|x_i\| = 1$. Consequently, we can take

$$y \in \left(\bigcap_{i=1}^n \bigcap_{f\in A_i} \ker(f) \cap \ker(x_i^*)\right) \cap S_X.$$

Pick $i \in \{1,\ldots,n\}$. We claim that

$$x_i \pm (\frac{r}{1+r} - \varepsilon)y \in S(B_{X^*}, x_i^*, \alpha_i).$$

First, notice that

$$x_i^*(x_i \pm (\frac{r}{1+r} - \varepsilon)y) = x_i^*(x_i) > 1 - \alpha$$

since $y \in \ker(x_i^*)$. On the other hand let us prove that $|||x_i \pm (\frac{r}{1+r} - \varepsilon)y||| \leq 1$. To this end, notice that

$$|||x_i \pm (\frac{r}{1+r} - \varepsilon)y||| = \sup_{f\in \text{ext}(B_{X^*})} |f(x_i \pm (\frac{r}{1+r} - \varepsilon)y)|.$$

Given $f \in \text{ext}(B_{X^*})$ we have two cases to consider:

Case 1: If $f \in A_i$ we get that $f(y) = 0$, and so

$$|f(x_i \pm (\frac{r}{1+r} - \varepsilon)y)| = |f(x_i)| \leq 1.$$

Case 2: If $f \notin A_i$, then $|f(x_i)| \leq \frac{1}{1+r} + \varepsilon$, and so

$$|f(x_i \pm (\frac{r}{1+r} - \varepsilon)y)| \leq |f(x_i)| + (\frac{r}{1+r} - \varepsilon)|f(y)| \leq \frac{1}{1+r} + \frac{r}{1+r} = 1.$$

This implies that $\sum_{i=1}^n \lambda_i(x_i \pm (\frac{r}{1+r} - \varepsilon)y) \in C$, so

$$\begin{aligned}\text{diam}(C) &\geq \left|\left|\left| \sum_{i=1}^n \lambda_i \left(x_i + (\frac{r}{1+r} - \varepsilon)y\right) - \sum_{i=1}^n \lambda_i \left(x_i - (\frac{r}{1+r} - \varepsilon)y\right) \right|\right|\right| \\ &= 2\frac{r}{1+r} - 2\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary we deduce that $\text{diam}(C) \geq \frac{2r}{1+r}$. Consequently, if there exists a convex combination of slices C such that $C \subseteq B(0, \rho)$, then

$$\frac{2r}{1+r} \leq \text{diam}(C) \leq \text{diam}(B(0,\rho)) = 2\rho \Rightarrow \rho \geq \frac{r}{1+r}.$$

This implies that $\mathcal{T}^{cc}(X) \geq \frac{r}{1+r}$.

Hence we have proved that

$$\frac{r}{1+r} \leq \mathcal{T}^{cc}(X) \leq \mathcal{T}(X) \leq \mathcal{T}^s(X) \leq \frac{r}{1+r},$$

as desired. □

Proof of Theorem 2.7. First observe that, if r equals $0, 1$ or 2, then take X to be ℓ_1, c_0 or $C[0,1]$, respectively. If $r \in (0,1)$, then there exists a $s \in (0,\infty)$ such that $r = \frac{s}{s+1}$ and apply Proposition 2.8 to s. If $r \in (1,2)$, then there exists a $p \in (1,\infty)$ such that $r = 2^{1/p}$ and apply Theorem 2.6 to $X = C[0,1] \oplus_p C[0,1]$. □

As a consequence of our Proposition 2.8 we obtain the following result.

Corollary 2.9. *For every $\delta \in (0,2)$ there exists a Banach space X such that every convex combination of slices has diameter $\geq \delta$ but such that, for every $\varepsilon > 0$, there exists a slice S of B_X such that* $\text{diam}(S) \leq \delta + \varepsilon$.

The previous corollary extends [10, Corollary 2.9], where it was proved for the cases $\delta \in (1,2)$, and answers an open question (see the Remark after Theorem 2.8 in [10]).

In Proposition 2.1 (a), it seems to be unknown whether the inequality can be strict (see Question 2.14). However, for the index $\mathcal{T}^s(\cdot)$, we always have equality.

Proposition 2.10. *Let X and Y be Banach spaces. Then*

(a) $\mathcal{T}^s(X \oplus_1 Y) = \min\{\mathcal{T}^s(X), \mathcal{T}^s(Y)\}$;
(b) $\mathcal{T}^s(X \oplus_p Y) \le 2^{1/p}$ for every $1 < p < \infty$;
(c) $\mathcal{T}^s(X \oplus_\infty Y) \ge \min\{\mathcal{T}^s(X), \mathcal{T}^s(Y)\}$, where equality holds if $\mathcal{T}^s(X \oplus_\infty Y) > 1$.

Proof. (a). Let us first show that $\mathcal{T}^s(X \oplus_1 Y) \ge \min\{\mathcal{T}^s(X), \mathcal{T}^s(Y)\}$. Set $Z := X \oplus_1 Y$ and let $S(B_Z, (x^*, y^*), \alpha)$ be a slice of B_Z, $(x, y) \in S_Z$, and $\varepsilon > 0$.

Without loss of generality suppose that $\|x^*\| = 1$. Thus we have two cases either $x = 0$ or $x \neq 0$.

Case 1: Assume first that $x = 0$, hence $\|y\| = 1$. Now find an element $u \in S_X$ such that $x^*(u) > 1 - \alpha$. Observe that $(u, 0) \in S(B_Z, (x^*, y^*), \alpha)$ and

$$\|(u,0) - (0,y)\| = 2 \ge \mathcal{T}^s(X).$$

Case 2: Assume now that $x \neq 0$. Consider the slice $S(B_X, x^*, \alpha)$ and $\frac{x}{\|x\|} \in S_X$. Find $u \in S(B_X, x^*, \alpha)$ such that $\|\frac{x}{\|x\|} - u\| \ge \mathcal{T}^s(X) - \varepsilon$. Now $(u, 0) \in S(B_Z, (x^*, y^*), \alpha)$ and

$$\begin{aligned}
\|(x,y) - (u,0)\| &= \|x - u\| + \|y\| \\
&\ge \|\frac{x}{\|x\|} - u\| - \|\frac{x}{\|x\|} - x\| + \|y\| \\
&\ge \mathcal{T}^s(X) - \varepsilon - (1 - \|x\|) + \|y\| \\
&= \mathcal{T}^s(X) - \varepsilon.
\end{aligned}$$

Therefore, $\mathcal{T}^s(X \oplus_1 Y) \ge \min\{\mathcal{T}^s(X), \mathcal{T}^s(Y)\}$.

The proof of $\mathcal{T}^s(X \oplus_1 Y) \le \min\{\mathcal{T}^s(X), \mathcal{T}^s(Y)\}$ is the same as the proof of [13, Proposition 4.5 (2)], except with $m = 1$.

(b). This follows immediately from Proposition 2.4.

(c). Let us first show that $\mathcal{T}^s(X \oplus_\infty Y) \ge \min\{\mathcal{T}^s(X), \mathcal{T}^s(Y)\}$. Denote $Z := X \oplus_\infty Y$. Let $S(B_Z, (x^*, y^*), \alpha)$ be a slice of B_Z, $(x, y) \in S_Z$, and $\varepsilon > 0$.

Define

$$S^X := \begin{cases} S(B_X, \dfrac{x^*}{\|x^*\|}, \alpha), & \text{if } x^* \neq 0, \\ B_X, & \text{if } x^* = 0, \end{cases}$$

and

$$S^Y := \begin{cases} S(B_Y, \dfrac{y^*}{\|y^*\|}, \alpha), & \text{if } y^* \neq 0, \\ B_Y, & \text{if } y^* = 0. \end{cases}$$

Observe that $S^X \times S^Y \subset S(B_Z, (x^*, y^*), \alpha)$. Since $\max\{\|x\|, \|y\|\} = 1$, we will suppose from now on that $\|x\| = 1$. Hence there exists an $x_0 \in S^X$ such that $\|x_0 - x\| > \mathcal{T}^s(X) - \varepsilon$. Let $y_0 \in S^Y$ be arbitrary. We have that

$$\|(x_0, y_0) - (x, y)\| = \max\{\|x_0 - x\|, \|y_0 - y\|\} \geq \mathcal{T}^s(X) - \varepsilon.$$

The case when $\|y\| = 1$ is similar. Therefore, by the arbitrariness of ε, we see that $\mathcal{T}^s(X \oplus_\infty Y) \geq \min\{\mathcal{T}^s(X), \mathcal{T}^s(Y)\}$.

Assume now that $\mathcal{T}^s(X \oplus_\infty Y) > 1$ and let us show that then $\mathcal{T}^s(X \oplus_\infty Y) \leq \min\{\mathcal{T}^s(X), \mathcal{T}^s(Y)\}$. Pick an $\varepsilon > 0$ such that $\mathcal{T}^s(X \oplus_\infty Y) - \varepsilon > 1$. Let $x \in S_X$ and $S(B_X, x^*, \alpha)$ be a slice of B_X. Observe that $S(B_Z, (x^*, 0), \alpha)$ is a slice of B_Z and $(x, 0) \in S_Z$. Thus there is an element $(u, v) \in S(B_Z, (x^*, 0), \alpha)$ such that

$$1 < \mathcal{T}^s(X \oplus_\infty Y) - \varepsilon \leq \|(x, 0) - (u, v)\| = \max\{\|x - u\|, \|v\|\}.$$

Since $\|v\| \leq 1$, we have that $\|x - u\| \geq \mathcal{T}^s(X \oplus_\infty Y) - \varepsilon$. Notice that $u \in S(B_X, x^*, \alpha)$, hence $\mathcal{T}^s(X) \geq \mathcal{T}^s(X \oplus_\infty Y) - \varepsilon$. Finally, by the arbitrariness of ε, we conclude that $\mathcal{T}^s(X \oplus_\infty Y) \leq \min\{\mathcal{T}^s(X), \mathcal{T}^s(Y)\}$. $\square$

Remark 2.11. The inequality in Proposition 2.10 (c) can be strict if we remove the assumption on $\mathcal{T}^s(X \oplus_\infty Y)$. Indeed, let $X = c_0$ and $Y = \mathbb{R}$, then $X \oplus_\infty Y$ is isometrically isomorphic to c_0 and

$$T^s(X \oplus_\infty Y) = 1 > 0 = \mathcal{T}^s(\mathbb{R}) = \min\{\mathcal{T}^s(X), \mathcal{T}^s(Y)\}.$$

We end this section by studying the index $\mathcal{T}^{cc}(\cdot)$ in ℓ_p-sums.

Proposition 2.12. *Let X and Y be Banach spaces. Then*

(a) $\mathcal{T}^{cc}(X \oplus_1 Y) \leq \min\{\mathcal{T}^{cc}(X), \mathcal{T}^{cc}(Y)\}$*;*
(b) $\mathcal{T}^{cc}(X \oplus_p Y) \leq 2^{1/p}$ *for every* $1 < p < \infty$*;*
(c) $\mathcal{T}^{cc}(X \oplus_\infty Y) \geq \min\{\mathcal{T}^{cc}(X), \mathcal{T}^{cc}(Y)\}$*, where equality holds if* $\mathcal{T}^{cc}(X \oplus_\infty Y) > 1$*.*

Proof. (a). Let $\varepsilon > 0$, and assume without loss of generality that $\min\{\mathcal{T}^{cc}(X), \mathcal{T}^{cc}(Y)\} = \mathcal{T}^{cc}(X)$. Find a convex combination of slices $\sum_{i=1}^n \lambda_i S(B_X, x_i^*, \alpha_i)$ of B_X and an $x \in S_X$ such that

$$\sum_{i=1}^n \lambda_i S(B_X, x_i^*, \alpha_i) \subset B(x, \mathcal{T}^{cc}(X) + \varepsilon).$$

Let $\delta \in \min_i(0, \alpha_i)$ and set $Z := X \oplus_1 Y$. Observe that

$$S(B_Z, (x_i^*, 0), \delta) \subset S(B_X, x_i^*, \alpha_i) \times \delta B_Y$$

for every $i \in \{1, \dots, n\}$. Therefore,

$$\begin{aligned}\sum_{i=1}^n \lambda_i S(B_Z, (x_i^*, 0), \delta) &\subset \sum_{i=1}^n \lambda_i S(B_X, x_i^*, \alpha_i) \times \delta B_Y \\ &\subset B(x, \mathcal{T}^{cc}(X) + \varepsilon) \times \delta B_Y \\ &\subset B((x, 0), \mathcal{T}^{cc}(X) + \varepsilon + \delta).\end{aligned}$$

Since ε and δ can be chosen to be arbitrarily small, we have that $\mathcal{T}^{cc}(X \oplus_1 Y) \leq \min\{\mathcal{T}^{cc}(X), \mathcal{T}^{cc}(Y)\}$.

(b). This follows immediately from the inequality $\mathcal{T}^{cc}(\cdot) \leq \mathcal{T}^{s}(\cdot)$ and Proposition 2.10 (b).

(c). Let us first show that $\mathcal{T}^{cc}(X \oplus_\infty Y) \geq \min\{\mathcal{T}^{cc}(X), \mathcal{T}^{cc}(Y)\}$. Denote $Z := X \oplus_\infty Y$. Let $n \in \mathbb{N}$, for every $i \in \{1, \dots, n\}$ let $S(B_Z, (x_i^*, y_i^*), \alpha_i)$ be slices of B_Z, $\lambda_i > 0$ with $\sum_{i=1}^n \lambda_i = 1$, $(x, y) \in S_Z$, and $\varepsilon > 0$. Denote $S := \sum_{i=1}^n \lambda_i S(B_Z, (x_i^*, y_i^*), \alpha_i)$.

Define

$$S_i^X := \begin{cases} S(B_X, \dfrac{x_i^*}{\|x_i^*\|}, \alpha_i), & \text{if } x_i^* \neq 0, \\ B_X, & \text{if } x_i^* = 0, \end{cases}$$

and

$$S_i^Y := \begin{cases} S(B_Y, \dfrac{y_i^*}{\|y_i^*\|}, \alpha_i), & \text{if } y_i^* \neq 0, \\ B_Y, & \text{if } y_i^* = 0. \end{cases}$$

Denote $S^X := \sum_{i=1}^n \lambda_i S_i^X$ and $S^Y := \sum_{i=1}^n \lambda_i S_i^Y$. Notice that $S_i^X \times S_i^Y \subset S(B_Z, (x_i^*, y_i^*), \alpha_i)$ and that therefore $S^X \times S^Y \subset S$. Since $\max\{\|x\|, \|y\|\} = 1$, we will suppose from now on that $\|x\| = 1$. Hence there exists an $x_0 \in S^X$ such that $\|x_0 - x\| > \mathcal{T}^{cc}(X) - \varepsilon$. Let $y_0 \in S^Y$ be arbitrary. We have that

$$\|(x_0, y_0) - (x, y)\| = \max\{\|x_0 - x\|, \|y_0 - y\|\} \geq \mathcal{T}^{cc}(X) - \varepsilon.$$

The case when $\|y\| = 1$ is similar. Therefore, by the arbitrariness of ε, we see that $\mathcal{T}^{cc}(X \oplus_\infty Y) \geq \min\{\mathcal{T}^{cc}(X), \mathcal{T}^{cc}(Y)\}$.

Assume now that $\mathcal{T}^{cc}(X \oplus_\infty Y) > 1$ and let us show that then $\mathcal{T}^{cc}(X \oplus_\infty Y) \leq \min\{\mathcal{T}^{cc}(X), \mathcal{T}^{cc}(Y)\}$. Pick an $\varepsilon > 0$ such that $\mathcal{T}^{cc}(X \oplus_\infty Y) - \varepsilon > 1$. Let $x \in S_X$, $S(B_X, x_i^*, \alpha_i)$ be slices of B_X, and $\lambda_i > 0$, such that $\sum_{i=1}^n \lambda_i = 1$. Observe that for each i we have that $S(B_Z,(x_i^*,0),\alpha_i)$ is a slice of B_Z and $(x,0)\in S_Z$. Thus there is an element $(u,v)\in\sum_{i=1}^n\lambda_i S(B_Z,(x_i^*,0),\alpha_i)$ such that

$$1<\mathcal{T}^{cc}(X\oplus_\infty Y)-\varepsilon\le\|(x,0)-(u,v)\|=\max\{\|x-u\|,\|v\|\}.$$

Since $\|v\|\le 1$, we must have that $\|x-u\|\ge\mathcal{T}^{cc}(X\oplus_\infty Y)-\varepsilon$. Notice that $u\in\sum_{i=1}^n\lambda_i S(B_X,x_i^*,\alpha_i)$, hence $\mathcal{T}^{cc}(X)\ge\mathcal{T}^{cc}(X\oplus_\infty Y)-\varepsilon$. Finally, from the arbitrariness of ε, we conclude that $\mathcal{T}^{cc}(X\oplus_\infty Y)\le\min\{\mathcal{T}^{cc}(X),\mathcal{T}^{cc}(Y)\}$. $\square$

Remark 2.13. The same example as in Remark 2.11 shows that the inequality in Proposition 2.12 (c) can be strict if we remove the assumption on $\mathcal{T}^{cc}(X\oplus_\infty Y)$.

Recall that from Proposition 2.10 (a) we know that $\mathcal{T}^s(X\oplus_1 Y)=\min\{\mathcal{T}^s(X),\mathcal{T}^s(Y)\}$ holds for all Banach spaces X and Y. However, we do not know whether the corresponding equalities hold for the indices $\mathcal{T}(\cdot)$ and $\mathcal{T}^{cc}(\cdot)$ too.

Question 2.14. *Let X and Y be Banach spaces.*

(a) $\mathcal{T}(X\oplus_1 Y)=\min\{\mathcal{T}(X),\mathcal{T}(Y)\}$*?*
(b) $\mathcal{T}^{cc}(X\oplus_1 Y)=\min\{\mathcal{T}^{cc}(X),\mathcal{T}^{cc}(Y)\}$*?*

3. Remarks and open questions

In a dual Banach space one can also consider the weak* versions of the Daugavet indices of thickness. In [13] the weak* version of $\mathcal{T}(\cdot)$, denoted by $\mathcal{T}_{w^*}(\cdot)$, was introduced. For a Banach space X we will also consider

$$\mathcal{T}^s_{w^*}(X^*)=\inf\left\{r>0\;\middle|\;\begin{array}{l}\text{there exist } x^*\in S_{X^*} \text{ and a weak}^* \text{ slice}\\ S \text{ of } B_{X^*} \text{ such that } S\subset B(x^*,r)\end{array}\right\}$$

and

$$\mathcal{T}^{cc}_{w^*}(X^*)=\inf\left\{r>0\;\middle|\;\begin{array}{l}\text{there exist } x^*\in S_{X^*} \text{ and a convex}\\ \text{combination } C \text{ of relatively weak}^* \text{ open}\\ \text{subsets of } B_{X^*} \text{ such that } \emptyset\neq C\subset B(x^*,r)\end{array}\right\}.$$

Clearly, for any Banach space X we have that

$$0\le\mathcal{T}^{cc}_{w^*}(X^*)\le\mathcal{T}_{w^*}(X^*)\le\mathcal{T}^s_{w^*}(X^*)\le 2, \tag{3.1}$$

and observe that

$$\mathcal{T}^s(X^{**})\le\mathcal{T}^s_{w^*}(X^{**})\le\mathcal{T}^s(X) \tag{3.2}$$

and

$$\mathcal{T}(X^{**}) \leq \mathcal{T}_{w^*}(X^{**}) \leq \mathcal{T}(X), \tag{3.3}$$

and

$$\mathcal{T}^{cc}(X^{**}) \leq \mathcal{T}^{cc}_{w^*}(X^{**}) \leq \mathcal{T}^{cc}(X). \tag{3.4}$$

Remark 3.1. Let us make some observations on the above indices:

(a) By considering the biduals of the Banach spaces which give us the strict inequalities between the regular indices and taking into account (3.2)–(3.4) one has that the inequalities in (3.1) can in general be strict.
(b) Given a dual Banach space X^*, the inequality $\mathcal{T}(X^*) \leq \mathcal{T}_{w^*}(X)$ may be strict. Indeed, if $C[0,1]$, then $\mathcal{T}^{cc}_{w^*}(X^*) = 2$ since X has the Daugavet property. However, $\mathcal{T}^s(X^*) = 0$ since B_{X^*} contains slices of arbitrarily small diameter. This shows that the first inequality of (3.2)–(3.4) can be strict.
(c) Again take $X = C[0,1]$. It satisfies that $\mathcal{T}^{cc}(X) = 2$ since X has the Daugavet property. However, $\mathcal{T}^s_{w^*}(X^{**}) < 2$ since X^* fails the Daugavet property. This shows that the second inequality of (3.2)–(3.4) can be strict.

In [13, Problem 5.3] it is wondered whether the equality

$$\inf\left\{\|T+I\| \;\middle|\; \begin{array}{l} T \in \mathcal{L}(X),\ \|T\| = 1, \text{ and} \\ T \text{ is weakly compact} \end{array}\right\} = \max\{\mathcal{T}(X), \mathcal{T}_{w^*}(X^*)\} \tag{3.5}$$

holds for every Banach space X. We will now show that equality (3.5) does not hold in general. We begin by observing that the proof of [13, Proposition 4.4] actually shows that the following proposition holds.

Proposition 3.2. *Let X be a Banach space. Then, for every norm one and weakly compact operator $T\colon X \to X$, it follows that*

$$\|T+I\| \geq \max\{\mathcal{T}^s(X), \mathcal{T}^s_{w^*}(X^*)\}.$$

By [2, Theorem 2.4], there exists an equivalent renorming Z of c_0 such that all slices of B_Z have diameter two and there are relatively weakly open subsets of B_Z with arbitrarily small diameter. Then $\mathcal{T}^s(Z) \geq 1$, but $\mathcal{T}(Z) = 0 = \mathcal{T}_{w^*}(Z^*)$ (notice that Z^* has the Radon–Nikodým property because it is isomorphic to ℓ_1, and the result follows from [6, Theorem 11.8]). Therefore, the equality (3.5) fails for this Banach space Z.

Question 3.3. *Does the equality*

$$\inf\left\{\|T+I\| \;\middle|\; \begin{array}{l} T\in\mathcal{L}(X),\ \|T\|=1, \\ \text{and } T \text{ is weakly compact}\end{array}\right\} = \max\{\mathcal{T}^s(X), \mathcal{T}^s_{w^*}(X^*)\}$$

hold for every Banach space X?

Our next aim is to show that the Daugavet index $\mathcal{T}^s(\cdot)$ behaves well with respect to the Banach–Mazur distance. Recall that this distance between two isomorphic Banach spaces X and Y is defined by

$$d(X,Y) := \inf\{\|L\|\|L^{-1}\| \colon L\colon X\to Y \text{ is an isomorphism}\}.$$

Proposition 3.4. *Let X be a Banach space and $r\in[0,2]$. If for every $\delta>0$ there exists a Banach space Y which is isomorphic to X such that $d(X,Y)<1+\delta$ with $\mathcal{T}^s(Y)=r$, then $\mathcal{T}^s(X)\geq r$.*

Proof. Let $S(B_X,x^*,\alpha)$ be a slice of B_X, $x\in S_X$, and $\varepsilon>0$. Let $\delta\in(0,\min\{\alpha,\varepsilon\})$.

Next find a Banach space Y with $\mathcal{T}^s(Y)=r$ such that $d(X,Y)<1+\delta$. Then there exist an isomorphism $L: X\to Y$ and such that $\|L\|=1$ and $\|L^{-1}\|<1+\delta$.

Consider now

$$y^* := \frac{(L^{-1})^*x^*}{\|(L^{-1})^*x^*\|}\in S_{Y^*} \quad \text{and} \quad y := \frac{Lx}{\|Lx\|}\in S_Y.$$

Since $\mathcal{T}^s(Y)=r$, we can find a $v\in S(B_Y,y^*,\delta^2)$ such that $\|v-y\|\geq r-\varepsilon$. Denote $u:=\frac{L^{-1}v}{(1+\delta)}$ and observe that $u\in S(B_X,x^*,\delta)\subset S(B_X,x^*,\alpha)$. Our aim now is to show that $\|u-x\|\geq r-3\varepsilon$. Indeed,

$$\begin{aligned} r-\varepsilon &\leq \|v-\frac{Lx}{\|Lx\|}\| \leq \|L^{-1}v-\frac{x}{\|Lx\|}\| \\ &\leq \|u-x\| + \|(1+\delta)u-u\| + \|x-\frac{x}{\|Lx\|}\| \\ &\leq \|u-x\|+\delta+\delta \\ &< \|u-x\|+2\varepsilon. \end{aligned}$$

Hence, $\|u-x\|\geq\mathcal{T}^s(Y)-3\varepsilon$ and from the arbitrariness of ε, we have $\mathcal{T}^s(X)\geq\mathcal{T}^s(Y)$. $\square$

An application of Proposition 3.4 together with Proposition 1.2 immediately gives us that the class of Banach spaces with the Daugavet property is closed with respect to the Banach–Mazur distance.

Corollary 3.5. *Let X be a Banach space. If for every $\delta > 0$ there exists a Banach space Y which is isomorphic to X such that $\mathrm{d}(X, Y) < 1+\delta$ and Y has the Daugavet property, then X also has the Daugavet property.*

We will finish by connecting the current work with some open questions related to a question of Ivakhno from 2006.

Recall that for a bounded set C of a Banach space X the *radius of C* is defined as

$$r(C) := \inf\{r > 0 : C \subseteq B(x, r) \text{ for some } x \in X\}.$$

A Banach space X is said to have the *r-big slice property* if every slice of B_X is of radius one [11]. Observe that the r-big slice property of a Banach space X implies that $\mathcal{T}^s(X) \geq 1$. Moreover, if every slice of B_X has diameter two, then X has the r-big slice property. Ivakhno asked if the converse is true [11, p. 96].

In view of Ivakhno's question, given a Banach space X, the following questions make sense:

(a) If $\mathcal{T}^s(X) \geq 1$, then does every slice of B_X have diameter two?
(b) If $\mathcal{T}(X) \geq 1$, then does every nonempty relatively weakly open subset of B_X have diameter two?
(c) If $\mathcal{T}^{cc}(X) \geq 1$, then does every convex combination of slices of B_X have diameter two?

A negative answer to (c) easily follows from our results, as the following remark shows.

Remark 3.6. Let Y be a Banach space with the Daugavet property. For $1 < p < \infty$ define $X := Y \oplus_p Y$. By Theorem 2.6 we have $\mathcal{T}^{cc}(X) = 2^{1/p} > 1$. But convex combinations of slices of B_X does not have diameter two since, by [10, Theorem 2.8], for every $\varepsilon > 0$ there is a nonempty convex combination of slices with diameter less than $2^{1/p} + \varepsilon$.

We end this paper by proving that the answer to Ivakhno's question (and henceforth, the answer to (a)) is negative. Indeed, we have the following result.

Theorem 3.7. *There exists a Banach space X with the r-big slice property (hence $\mathcal{T}^s(X) \geq 1$) and there exists $\varphi \in S_{X^*}$ so that*

$$\inf_{\alpha>0} \operatorname{diam}(S(B_X, \varphi, \alpha)) \leq \sqrt{2}.$$

In order to prove it let us introduce a bit of notation. Let us define

$$T := \{(\alpha_1, \dots, \alpha_k) \ : \ k \in \mathbb{N}, \alpha_1, \dots, \alpha_k \in \mathbb{N}\} \cup \{\emptyset\}.$$

Given $(\alpha_1, \dots, \alpha_k), (\beta_1, \dots, \beta_p) \in T \setminus \{\emptyset\}$ we define

$$(\alpha_1, \ldots, \alpha_k) \leq (\beta_1, \ldots, \beta_p) \Leftrightarrow (k \leq p \text{ and } \alpha_i = \beta_i \text{ for } 1 \leq i \leq k)$$

and we declare $\emptyset \leq t$ for all $t \in T$. This binary relation defines a partial order on T. We say that t is an *immediate successor* of s if $s < t$ and the set $\{r \in T : s < r < t\}$ is empty. A *segment* in T is a totally ordered and finite subset $S \subseteq T$. Given $s = (\alpha_1, \ldots, \alpha_k)$ and $t = (\beta_1, \ldots, \beta_p)$ we define the *concatenation* of s and t by $s^\frown t = \{\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_p\}$.

Given $x : T \longrightarrow \mathbb{R}$, let us consider

$$\|x\| = \sup \left(\sum_{i=1}^{n} \left(\sum_{t \in S_i} x(t) \right)^2 \right)^{\frac{1}{2}},$$

where the sup is taken over all families $\{S_1, \ldots, S_n\}$ of disjoint segments of T.

Now JT_∞ is defined as the completion of the space of finitely nonzero functions defined on T (i.e. functions $x : T \longrightarrow \mathbb{R}$ such that $\{t \in T \ : \ x(t) \neq 0\}$ is finite) with respect to the above norm. Given $\alpha \in T$ let us define

$$e_\alpha(\beta) := \begin{cases} 1 & \text{if } \beta = \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Then it is known that $\{e_\alpha\}_{\alpha \in T}$ is a (countable) Markushevich basis for JT_∞ and that JT_∞ is a dual space. We denote by $\{e_\alpha^*\}_{\alpha \in T}$ the biorthogonal functionals of $\{e_\alpha\}_{\alpha \in T}$. Then $B_\infty := \overline{\operatorname{span}}\{e_\alpha^* : \alpha \in T\}$, where the closure is taken in JT_∞^*, is a complete predual of JT_∞.

The space JT_∞ was introduced in [8], where it is proved that B_∞ fails the Radon–Nikodým property. Furthermore, every infinite-dimensional subspace of JT_∞ contains an isomorphic copy of ℓ_2 and so JT_∞ does not contain isomorphic copies of ℓ_1.

Let us start with the following lemma about weakly null sequences in JT_∞.

Lemma 3.8. *Let $t \in T$ and define $t_n = t^\frown\{n\}$ for $n \in N$. Then $\{e_{t_n}\}_n$ is weakly null.*

Proof. Let $x := \sum_{j=1}^m \alpha_j e_{t_j}$, where $\alpha_j \in \mathbb{R}$, and let us prove that $\|x\| \leq \left(\sum_{j=1}^m \alpha_j^2\right)^{\frac{1}{2}}$. To this end, pick a family of disjoint segments $S_1, \ldots, S_k$. Observe that, for every $i \in \{1, \ldots, k\}$, $S_i \cap \{t_1, \ldots, t_m\}$ has at most one element, because $\{t_n\}$ are incomparable. Define A to be the set of those i such that $S_i \cap \{t_1, \ldots, t_m\} = \{t_{k_i}\}$. Notice that, since the S_i are disjoint, $t_{k_i} \neq t_{k_j}$ if $i \neq j$ with $i, j \in A$. Now

$$\sum_{i=1}^{m} \left(\left(\sum_{t \in S_i} x(t) \right)^2 \right)^{\frac{1}{2}} = \left(\sum_{i \in A} \alpha_{t_{k_i}}^2 \right)^{\frac{1}{2}} \leq \left(\sum_{i=1}^{m} \alpha_i^2 \right)^{\frac{1}{2}}.$$

Taking the supremum over the family of disjoint segments, and taking into account the definition of the norm of JT_∞, we get

$$\|x\| \leq \left(\sum_{i=1}^{m} \alpha_i^2\right)^{\frac{1}{2}}.$$

The previous estimate implies, by the arbitrariness of $m \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$, that the linear operator $\Phi : \ell_2 \longrightarrow JT_\infty$ given by

$$\Phi(e_n) = e_{t_n}$$

is continuous. The $w - w$ continuity of Φ and the fact that $\{e_n\} \to^w 0$ in ℓ_2 concludes the lemma. □

Now we are ready to prove Theorem 3.7.

Proof of Theorem 3.7. Let $X = B_\infty$, the predual of JT_∞ described above. The existence of $\varphi \in S_{X^*}$ satisfying our requirements follows from [3, Theorem 2.2]. For the remaining part, let us even prove that, given a w^*-slice S of $B_{X^{**}} = B_{JT_\infty^*}$, we get that

$$r(S) \geq 1.$$

To this end, pick a w^*-slice $S := S(B_{JT_\infty^*}, x, \alpha)$, for a suitable finitely-supported function $x : T \longrightarrow \mathbb{R}$ of norm one. Pick $x^* \in JT_\infty^*$ and $\varepsilon > 0$, and let us find an element $y^* \in S$ with $\|x^* - y^*\| \geq 1 - \varepsilon$. To this end, by [3, Lemma 2.1] we can find an element $g \in S$ of the form $g := \sum_{i=1}^n \lambda_i f_{S_i}$, where $\lambda_1, \ldots, \lambda_n \in \mathbb{R}^+$ with $\sum_{i=1}^n \lambda_i^2 = 1$ and $S_1, \ldots, S_n$ is a family of disjoint segments.

Since the set of immediate successors of a given element is infinite and the fact that the support of x is finite, we can assume (by adding elements which do not belong to $\operatorname{supp}(x)$ into S_i keeping the disjointness condition on the segments $S_1, \ldots, S_n$) that, if t_i is the maximum element of S_i, then for every $z \geq t_i$ one has $z \notin \operatorname{supp}(x)$ and that $\{t_1, \ldots, t_n\}$ are at the same level.

For each $i \in \{1, \ldots, n\}$ consider $t_{i,m} := t_i \frown \{m\}$ for $m \in \mathbb{N}$. By Lemma 3.8 $\{e_{t_{i,m}}\}_m$ is weakly null, hence $x^*(e_{t_{i,m}}) \to_m 0$. Denote $\lambda := \min_{1 \leq i \leq n} \lambda_i$. Find k large enough so that $x^*(e_{t_{i,k}}) < \frac{\varepsilon}{\lambda}$ holds for every $i \in \{1, \ldots, n\}$. Define $R_i := S_i \cup \{t_{i,k}\}$ and notice that $y^* := \sum_{i=1}^n \lambda_i f_{R_i}$ is a norm-one element (because $\{R_1, \ldots, R_n\}$ is still a family of disjoint segments) and that $y^* \in S$ (indeed, notice that $y^*(x) = g(x)$ because $t_{i,k} \notin \operatorname{supp}(x)$ for every i). Define $z : T \longrightarrow \mathbb{R}$ by $z = \sum_{i=1}^n \lambda_i e_{t_{i,k}}$. Notice that $y^*(z) = \sum_{i=1}^n \lambda_i^2 = 1$ by assumptions. Moreover, similar estimates to the ones of Lemma 3.8 prove that $\|z\| \leq 1$ in JT_∞. Moreover

$$x^*(z) = \sum_{i=1}^{n} \lambda_i x^*(e_{t_{i,k}}) < \varepsilon.$$

So

$$\|y^* - x^*\| \geq (y^* - x^*)(z) > 1 - \varepsilon,$$

as desired. □

Remark 3.9. Notice that $\mathcal{T}(B_\infty) = 0$ since the unit ball of B_∞ contains nonempty relatively weakly open subsets of arbitrarily small diameter (in fact, B_∞ has the *convex point of continuity property* [9, Theorem 2.2]). Consequently, question (b) remains open.

Acknowledgment

The authors wish to thank the anonymous referee for a careful reading of the manuscript that helped to improve the presentation.

References

[1] F.F. Bonsall, J. Duncan, Numerical Ranges II, London Mathematical Society Lecture Note Series, vol. 10, Cambridge University Press, 1973.
[2] J. Becerra Guerrero, G. López-Pérez, A. Rueda Zoca, Big slices versus big relatively weakly open subsets in Banach spaces, J. Math. Anal. Appl. 428 (2) (2015) 855–865.
[3] J. Becerra Guerrero, G. López-Pérez, A. Rueda Zoca, Diameter two properties in James spaces, Banach J. Math. Anal. 9 (4) (2015) 203–220.
[4] J. Becerra Guerrero, G. López-Pérez, A. Rueda Zoca, Extreme differences between weakly open subsets and convex combinations of slices in Banach spaces, Adv. Math. 269 (2015) 56–70.
[5] J. Becerra Guerrero, G. López-Pérez, A. Rueda Zoca, Diametral diameter two properties in Banach spaces, J. Convex Anal. 25 (3) (2018) 817–840.
[6] M. Fabian, P. Habala, P. Hájek, V. Montesinos, V. Zizler, Banach Space Theory, Springer Science+Business Media, LLC, 2011.
[7] N. Ghoussoub, G. Godefroy, B. Maurey, W. Schachermayer, Some topological and geometrical structures in Banach spaces, Mem. Am. Math. Soc. 70 (1987) 378, iv+116.
[8] N. Ghoussoub, B. Maurey, G_δ-embeddings in Hilbert space, J. Funct. Anal. 61 (1) (1985) 72–97.
[9] N. Ghoussoub, B. Maurey, W. Schachermayer, A counterexample to a problem on points of continuity in Banach spaces, Proc. Am. Math. Soc. 99 (2) (1987) 278–282.
[10] R. Haller, J. Langemets, R. Nadel, Stability of average roughness, octahedrality, and strong diameter 2 properties of Banach spaces with respect to absolute sums, Banach J. Math. Anal. 12 (1) (2018) 222–239.
[11] Y. Ivakhno, On sets with extremely big slices, J. Math. Phys. Anal. Geom. 2 (1) (2006) 94–103.
[12] G. López-Pérez, A. Rueda Zoca, Diameter two properties and polyhedrality, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. 113 (2019) 131–135.
[13] A. Rueda Zoca, Daugavet property and separability in Banach spaces, Banach J. Math. Anal. 12 (1) (2018) 68–84.
[14] A. Rueda Zoca, P. Tradacete, I. Villanueva, Daugavet property in tensor product spaces, J. Inst. Math. Jussieu (2019) 1–20, https://doi.org/10.1017/S147474801900063X.
[15] R.V. Shvydkoy, Geometric aspects of the Daugavet property, J. Funct. Anal. 176 (2) (2000) 198–212.
[16] D. Werner, Recent progress on the Daugavet property, Ir. Math. Soc. Bull. 46 (2001) 77–79.
[17] R. Whitley, The size of the unit sphere, Can. J. Math. 20 (1968) 450–455.
[18] P. Wojtaszczyk, Some remarks on the Daugavet equation, Proc. Am. Math. Soc. 115 (1992) 1047–1052.