Characterisation of the weak-star symmetric strong diameter 2 property in Lipschitz spaces

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Abstract

We give a characterisation of the weak∗ symmetric strong diameter 2 property for Lipschitz function spaces in terms of a property of the corresponding metric space. Using this characterisation we show that the weak∗ symmetric strong diameter 2 property is different from the weak∗ strong diameter 2 property in Lipschitz spaces, thereby answering a question posed in a recent paper by Haller, Langemets, Lima, and Nadel.

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1. Introduction

We consider only real Banach spaces. We start by fixing some notation. Given a metric space M and a point x in M, we denote by B(x, r) the open ball in M centred at x of radius r. Let X be a Banach space. We denote the closed unit ball, the unit sphere, and the dual space of X by BX, SX, and X∗, respectively. A weak∗ slice of BX∗ is a set of the form

\[ S(B_{X^*}, x, \alpha) := \{ x^* \in B_{X^*} : \langle x, x^* \rangle > 1 - \alpha \}, \]

where x ∈ SX and α > 0.

Let M be a pointed metric space, that is, a metric space with a fixed point 0. The space Lip0(M) of all Lipschitz functions f: M → R with f(0) = 0 is a Banach space with the norm

\[ \|f\|_{\text{Lip}} = \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} : x, y \in M, x \neq y \right\}. \]

The space

\[ \mathcal{F}(M) := \text{span} \{ \delta_m : m \in M \} \subset \text{Lip}_0(M)^* \]

is called the Lipschitz-free space over M, where \( \delta_m : \text{Lip}_0(M) \to \mathbb{R} \),

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\begin{equation}
\langle f, \delta_m \rangle = f(m), \quad m \in M, \ f \in \text{Lip}_0(M).
\end{equation}

It can be shown that, under this duality, $\mathcal{F}(M)^*$ is isometrically isomorphic to $\text{Lip}_0(M)$.

Recall that the dual space $X^*$ is said to have the weak* strong diameter 2 property ($w^*$-SD2P) if every finite convex combination of weak* slices of $B_{X^*}$ has diameter 2. It is well known that $X^*$ has the $w^*$-SD2P if and only if the norm of $X$ is octahedral ([5],[6], for a proof, see, e.g., [3] or [8]). Therefore, $\text{Lip}_0(M)$ has the $w^*$-SD2P if and only if the norm of $\mathcal{F}(M)$ is octahedral. Moreover, in [10, Theorem 3.1], it was shown that the norm of $\mathcal{F}(M)$ is octahedral if and only if the metric space $M$ has the following property.

**Definition 1.1.** A metric space $M$ is said to have the long trapezoid property (LTP) if, for every finite subset $N$ of $M$ and $\varepsilon > 0$, there exist $u, v \in M$, $u \neq v$, such that, for any $x, y \in N$,

\begin{equation}
(1 - \varepsilon)(d(x, y) + d(u, v)) \leq d(x, u) + d(y, v).
\end{equation}

Therefore, the Lipschitz space $\text{Lip}_0(M)$ has the $w^*$-SD2P if and only if $M$ has the LTP. The objective of this paper is to give a similar characterisation to the following property, which was introduced in [1] but studied more extensively in [2], [7], [4], and [9].

**Definition 1.2.** A dual Banach space $X^*$ is said to have the weak* symmetric strong diameter 2 property ($w^*$-SSD2P) if, for every finite family $\{S_i\}_{i=1}^n$ of weak* slices of $B_{X^*}$ and $\varepsilon > 0$, there exist $f_i \in S_i$, $i = 1, \ldots, n$, and $g \in B_{X^*}$ such that $f_i \pm g \in S_i$ for every $i \in \{1, \ldots, n\}$ and $\|g\| > 1 - \varepsilon$.

It is known that in general the $w^*$-SSD2P is a strictly stronger property than the $w^*$-SD2P (see, e.g., [7]). In this paper, we show that the same is true for Lipschitz function spaces, thus giving an answer to [7, Question 6.3].

The paper is organised as follows.

In Section 2, we give a characterisation of the $w^*$-SSD2P for the Lipschitz space $\text{Lip}_0(M)$ in terms of a property of the metric space $M$. More precisely, we prove Theorem 2.1, which says that $\text{Lip}_0(M)$ has the $w^*$-SSD2P if and only if $M$ enjoys the following property.

**Definition 1.3.** We say that $M$ has the strong long trapezoid property (SLTP) if, for every finite subset $N$ of $M$ and $\varepsilon > 0$, there exist $u, v \in M$, $u \neq v$, such that, for any $x, y \in N$, the inequality (1.1) holds, and, for any $x, y, z, w \in N$,

\begin{equation}
(1 - \varepsilon)(2d(u, v) + d(x, y) + d(z, w)) \leq d(x, u) + d(y, u) + d(z, v) + d(w, v).
\end{equation}

In Section 3, we first apply Theorem 2.1 to show that, for Lipschitz spaces, the $w^*$-SSD2P is a strictly stronger property than the $w^*$-SD2P: Example 3.1 provides a metric space which has the LTP but not the SLTP.

A question that arises from the definition of the SLTP is whether the inequality (1.2) implies (1.1). Example 3.2 shows that this is not the case: it provides a metric space $M$ for which (1.2) holds for every finite subset $N$, but which fails the LTP.

We finish the paper by showing that any infinite subset of $\ell_1$, viewed as a metric space, has the SLTP (Example 3.3).

2. Main result

**Theorem 2.1.** Let $M$ be a pointed metric space. The following statements are equivalent:
(i) $\text{Lip}_0(M)$ has the $w^*$-SSD2P;

(ii) $M$ has the SLTP.

**Proof.** (i)$\Rightarrow$(ii). Assume that $\text{Lip}_0(M)$ has the $w^*$-SSD2P, and let $N$ be a finite subset of $M$ and $0 < \varepsilon < 1$. Choose $\alpha > 0$ such that $2\alpha < \varepsilon$ and, for any $x, y \in N$, $x \neq y$,

$$\alpha \leq \frac{1}{d(x, y)} \quad \text{and} \quad 2\alpha \leq d(x, y).$$

For any $x, y \in N$, $x \neq y$, define a slice $S_{x,y} := S\left(B_{\text{Lip}_0(M)}, \frac{\delta_x - \delta_y}{d(x,y)}, \alpha^3\right)$. Since $\text{Lip}_0(M)$ has the $w^*$-SSD2P, we can find $f_{x,y} \in S_{x,y}$ and $g \in B_{\text{Lip}_0(M)}$, $\|g\| \geq 1 - \alpha$, such that $\|f_{x,y} \pm g\| \leq 1$. For $x, y \in N$, $x = y$, define $f_{x,y} := 0 \in \text{Lip}_0(M)$.

For any $x, y \in N$,

$$(f_{x,y}, \delta_x - \delta_y) = f_{x,y}(x) - f_{x,y}(y) \geq (1 - \alpha^3)d(x, y),$$

therefore, keeping in mind that $\|f_{x,y} \pm g\| \leq 1$,

$$\|g\| = |g(x) - g(y)| \leq \alpha^3d(x, y) \leq \alpha^2.$$

Since $\|g\| \geq 1 - \alpha$, there exist $u, v \in M$, $u \neq v$, such that

$$(g, \delta_u - \delta_v) = g(u) - g(v) \geq (1 - \alpha) d(u, v).$$

Now, for any $x, y \in N$, again using that $\|f_{x,y} \pm g\| \leq 1$,

$$\|(f_{x,y}, \delta_u - \delta_v)\| = |f_{x,y}(u) - f_{x,y}(v)| \leq \alpha d(u, v).$$

Letting $x, y, z, w \in N$ be arbitrary, it remains to verify (1.1) and (1.2). Since $\|f_{x,y} \pm g\| \leq 1$, we get

$$(1 - \varepsilon)(d(u, v) + d(x, y))$$

$$\leq (1 - 2\alpha)d(u, v) + (1 - 2\alpha^3)d(x, y)$$

$$\leq \langle g, \delta_u - \delta_v \rangle - \langle f_{x,y}, \delta_u - \delta_v \rangle + \langle f_{x,y}, \delta_x - \delta_y \rangle - \langle g, \delta_x - \delta_y \rangle$$

$$= \langle g - f_{x,y}, \delta_u - \delta_x \rangle - \langle g - f_{x,y}, \delta_v - \delta_y \rangle$$

$$\leq d(x, u) + d(y, v).$$

Thus, (1.1) holds. If $x = y$ and $z = w$, then (1.2) follows from (1.1) with $y$ replaced by $z$. If $x \neq y$ or $z \neq w$, then

$$\alpha (d(x, y) + d(z, w)) \geq 2\alpha^2 \geq |\langle g, \delta_z - \delta_x + \delta_w - \delta_y \rangle|,$$

and thus, since $\|f_{x,y} \pm g\| \leq 1$,

$$(1 - \varepsilon)(2d(u, v) + d(x, y) + d(z, w))$$

$$\leq 2(g(u) - g(v)) + (1 - \alpha^3 - \alpha)(d(x, y) + d(z, w))$$

$$\leq 2\langle g, \delta_u - \delta_v \rangle + \langle f_{x,y}, \delta_x - \delta_y \rangle + \langle f_{z,w}, \delta_z - \delta_w \rangle$$

$$+ \langle g, \delta_z - \delta_x + \delta_w - \delta_y \rangle$$
Whenever $\approx i$ with weak $4$ and $\in \{\mathbb{R}\}$ Set $(ii)$ Let $\quad \sum_{j=1}^{n_i} \lambda_{ij} \delta_{x_{ij}}$ for some $n_i \in \mathbb{N}$, $\lambda_{ij} \in \mathbb{R} \setminus \{0\}$, and $x_{ij} \in M$, $j = 1, \ldots, n_i$. Now $N := \{0\} \cup \bigcup_{i=1}^{n} \{x_{i1}, \ldots, x_{im}\}$ is a finite subset of $M$. We may also assume that $\varepsilon < \min_{1 \leq i \leq n} \alpha_i$. This enables, for every $i \in \{1, \ldots, n\}$, to pick an $h_i \in S_i$ with $\|h_i\| < 1 - \varepsilon$.

By the SLTP, there exist $u, v \in M$, $u \neq v$, satisfying (1.1) and (1.2) for all $x, y, z, w \in N$. Setting

$$r_0 := \frac{1}{2} \min_{x, y \in N} (d(x, u) + d(y, u) - (1 - \varepsilon)d(x, y))$$

and

$$s_0 := \frac{1}{2} \min_{z, w \in N} (d(z, v) + d(w, v) - (1 - \varepsilon)d(z, w)),$$

one has $r_0 + s_0 \geq (1 - \varepsilon)d(u, v)$. Thus, there exist $r, s \geq 0$ with $r \leq r_0$ and $s \leq s_0$ such that

$$r + s = (1 - \varepsilon)^2d(u, v).$$

We may assume that $r > 0$. Define a function $g: M \to \mathbb{R}$ by

$$g(x) := \begin{cases} r - d(x, u) & \text{if } x \in B(u, r); \\ -s + d(x, v) & \text{if } x \in B(v, s); \\ 0 & \text{otherwise} \end{cases}$$

(we use the convention $B(v, s) = \emptyset$ if $s = 0$). Observe that $\|g\| \leq 1$ (here we use that, whenever $x \in B(u, r)$ and $y \in B(v, s)$, one has $g(y) \leq 0 \leq g(x)$, and thus $\|g(x) - g(y)\| = g(x) - g(y)$). One also has $\|g\| \geq (1 - \varepsilon)^2$, because

$$|g(u) - g(v)| = g(u) - g(v) = r + s = (1 - \varepsilon)^2d(u, v).$$

Set $L := N \cup B$ where $B := B(u, r) \cup B(v, s)$. We next show that, for every $i \in \{1, \ldots, n\}$, there is a $c_i \in \mathbb{R}$ such that, defining a function $f_i: L \to \mathbb{R}$ by $f_i|_{N} = h_i|_{N}$ and $f_i|_{B} = c_i$ (observe that $B \cap N = \emptyset$), one has $\|f_i \pm g\|_{\text{Lip}_{0}(L)} \leq 1$ and $\|f_i \pm g\|_{\text{Lip}_{0}(L)} \leq 1$.

Let $i \in \{1, \ldots, n\}$. Set

$$\tilde{a}_i := \max_{x \in N} (h_i(x) - d(x, u)), \quad \tilde{a}_i := \min_{x \in N} (h_i(x) + d(x, u)),$$

$$\tilde{b}_i := \max_{x \in N} (h_i(x) - d(x, v)), \quad \tilde{b}_i := \min_{x \in N} (h_i(x) + d(x, v)).$$

Whenever $x, y \in N$, since $\|h_i\| < 1 - \varepsilon$, one has

$$h_i(x) + d(x, u) - (h_i(y) - d(y, u)) \geq d(x, u) + d(y, u) - (1 - \varepsilon)d(x, y) \geq 2r,$$
and, by (1.1),

\[ h_i(x) + d(x, u) - (h_i(y) - d(y, v)) \geq d(x, u) + d(y, v) - (1 - \varepsilon)d(x, y) \]
\[ \geq (1 - \varepsilon)d(u, v) > r + s. \]

Thus, \( \hat{a}_i - r \geq \bar{a}_i + r \) and \( \hat{a}_i - r > \bar{b}_i + s \). Similarly, one observes that \( \hat{b}_i - s \geq \bar{b}_i + s \) and \( \hat{b}_i - s > \bar{a}_i + r \). It follows that there exists a \( c_i \in [\bar{a}_i + r, \hat{a}_i - r] \cap [\hat{b}_i + s, \bar{b}_i - s] \). This \( c_i \) does the job.

Indeed, let \( x \in N \) and \( y \in B(u, r) \). In order to see that

\[ |f_i(x) + g(x) - (f_i(y) + g(y))| = |h_i(x) - (c_i \pm (r - d(y, u)))| \leq d(x, y), \]

it suffices to show that

\[ h_i(x) - d(x, y) + d(y, u) \leq c_i \pm r \leq h_i(x) + d(x, y) \pm d(y, u). \]

These inequalities hold:

\[ h_i(x) - d(x, y) + d(y, u) \leq h_i(x) - d(x, u) \]
\[ \leq \bar{a}_i \leq c_i - r \leq \hat{a}_i - 2r \]
\[ \leq h_i(x) + d(x, u) - 2d(y, u) \]
\[ \leq h_i(x) + d(x, y) - d(y, u) \]

and

\[ h_i(x) - d(x, y) + d(y, u) \leq h_i(x) - d(x, u) + 2d(y, u) \]
\[ \leq \bar{a}_i + 2r \leq c_i + r \leq \hat{a}_i \]
\[ \leq h_i(x) + d(x, u) \]
\[ \leq h_i(x) + d(x, y) + d(y, u). \]

The inequalities

\[ |f_i(x) \pm |g(x)| - (f_i(y) \pm |g(y)|)| = |h_i(x) - (c_i \pm (r - d(y, u)))| \leq d(x, y) \]

follow from (2.1).

For every \( i \in \{1, \ldots, n\} \), we extend \( f_i \) to the entire space \( M \) by setting

\[ f_i(y) := \sup_{x \in L} (f_i(x) + |g(x)| - d(x, y)) \text{ for every } y \in M \setminus L. \]

Note that, on \( M \setminus L \), the function \( f_i \) agrees with a norm preserving extension of \( (f_i + g)|_{L} \). It remains to show that \( \|f_i + g\|_{\text{Lip}_0(M)} \leq 1 \). Indeed, this implies that also \( \|f_i\|_{\text{Lip}_0(M)} \leq 1 \), and thus \( f_i \in S_i \), because, since \( f_i|_N = h_i|_N \), one has \( \langle \mu_i, f_i \rangle = \langle \mu_i, h_i \rangle > 1 - \alpha_i \).

Let \( i \in \{1, \ldots, n\} \). To see that \( \|f_i + g\|_{\text{Lip}_0(M)} \leq 1 \), it suffices to show that, whenever \( x, y \in M \), one has

\[ -d(x, y) \leq f_i(x) \pm g(x) - (f_i(y) \pm g(y)) \leq d(x, y). \]

For the cases when \( x, y \in L \) or \( x, y \in M \setminus L \), or \( x \in N \) (or \( y \in N \)) and \( y \in M \setminus L \) (or \( x \in M \setminus L \)), the inequalities (2.2) follow from what has been proven above. So, in fact, it suffices to consider the case when \( x \in B(u, r) \cup B(v, s) \) and \( y \in M \setminus L \). In this case, (2.2) means that
\[ -d(x, y) \leq c_i \pm g(x) - \sup_{z \in L} \left( f_i(z) + |g(z)| - d(z, y) \right) \leq d(x, y). \]

Thus, it suffices to show that

1. there is a \( z \in L \) such that
   \[ c_i \pm g(x) - d(x, y) + d(z, y) \leq f_i(z) + |g(z)|; \]
2. for every \( z \in L \),
   \[ f_i(z) + |g(z)| \leq c_i \pm g(x) + d(x, y) + d(z, y). \]

For (1), one may take \( z = x \), so it remains to prove (2). By symmetry, it suffices to consider only the case when \( x \in B(u, r) \). In this case \( g(x) = r - d(x, u) \geq 0 \). Thus, it suffices to prove that, for every \( z \in L \),

\[ f_i(z) + |g(z)| \leq c_i - r + d(x, u) + d(x, y) + d(z, y). \]

One has to look through the following cases:

(a) \( z \in B(u, r); \)  
(b) \( z \in B(v, s); \)  
(c) \( z \in N. \)

(a). If \( z \in B(u, r) \), then \( f_i(z) = c_i \) and \( |g(z)| = r - d(z, u) \). Thus, one has to show that

\[ 2r \leq d(x, u) + d(z, u) + d(x, y) + d(z, y). \]

This inequality holds, because, since \( y \notin B(u, r) \), one has \( d(y, u) \geq r \), and thus

\[ 2r \leq d(y, u) + d(y, u) \leq d(x, u) + d(x, y) + d(z, u) + d(z, y). \]

(b). If \( z \in B(v, s) \), then \( f_i(z) = c_i \) and \( |g(z)| = s - d(z, v) \). Thus, one has to show that

\[ r + s \leq d(x, u) + d(z, v) + d(x, y) + d(z, y). \]

This inequality holds, because, since \( y \notin B(u, r) \) and \( y \notin B(v, s) \), one has \( d(y, u) \geq r \) and \( d(y, v) \geq s \), and thus

\[ r + s \leq d(y, u) + d(y, v) \leq d(x, u) + d(x, y) + d(z, v) + d(z, y). \]

(c). If \( z \in N \), then

\[ f_i(z) + |g(z)| = f_i(z) = h_i(z) = \alpha_i + d(z, u) \leq c_i - r + d(x, u) + d(x, y) + d(z, y). \]  \( \square \)

3. Examples

We now give an example of a metric space \( M \) that has the LTP but fails the SLTP. By [10, Theorem 3.1] and Theorem 2.1, this implies that the corresponding Lipschitz space \( \text{Lip}_0(M) \) has the \( w^* \)-SD2P but fails the \( w^* \)-SSD2P.
Fig. 1. A representation of the metric space $M$ in Example 3.1. The distances between points connected by a straight line segment are 1, the distances between other different points are 2.

**Example 3.1.** Let $M = \{a_1, a_2, b_1, b_2\} \cup \{u_i, v_i : i \in \mathbb{N}\}$ be a metric space where the distances between different points are defined as follows: for any $i \in \{1, 2\}$, $j, k, l \in \mathbb{N}$, $k \neq l$,

$$
d(a_1, a_2) = d(b_1, b_2) = d(a_i, v_j) = d(b_i, u_j) = d(u_k, u_l) = d(v_k, v_l) = d(u_k, v_l) = 2
$$

and, for any $i, j \in \{1, 2\}$, $k \in \mathbb{N}$,

$$
d(a_i, b_j) = d(a_i, u_k) = d(b_i, v_k) = d(u_k, v_k) = 1.
$$

(See Fig. 1.) We first show that $M$ has the LTP. Letting $N$ be a finite subset of $M$ and $i \in \mathbb{N}$ be such that $u_i, v_i \in M \setminus N$, it suffices to show that, for any $x, y \in N$,

$$
d(x, y) + d(u_i, v_i) = d(x, y) + 1 \leq d(x, u_i) + d(y, v_i).
$$

To this end, letting $x, y \in M \setminus \{u_i, v_i\}$ be such that $d(x, y) = 2$, it suffices to show that

$$
d(x, u_i) + d(y, v_i) \geq 3.
$$

For this, notice that if $x \in \{a_1, a_2\}$, then either $y \in \{a_1, a_2\}$ or $y \in \{v_j : j \in \mathbb{N} \setminus \{i\}\}$, but in both of these cases $d(y, v_i) = 2$ and $d(x, u_i) = 1$; if $x \in \{b_1, b_2\} \cup \{u_j, v_j : j \in \mathbb{N} \setminus \{i\}\}$, then $d(x, u_i) = 2$ and $d(y, v_i) \geq 1$.

It remains to show that $M$ fails the SLTP. Take $N := \{a_1, a_2, b_1, b_2\}$. Then, for any $u, v \in M$, $u \neq v$, there exist $x, y, z, w \in N$ such that

$$
2d(u, v) + d(x, y) + d(z, w) \geq d(x, u) + d(y, u) + d(z, v) + d(w, v) + 1.
$$

Indeed, set $U := \{u_i : i \in \mathbb{N}\}$ and $V := \{v_i : i \in \mathbb{N}\}$, and suppose that $u, v \in M$, $u \neq v$. If $u, v \in U$ or $u, v \in V$, then, respectively, for $x = z = a_1$, $y = w = a_2$, and for $x = z = b_1$, $y = w = b_2$,

$$
2d(u, v) + d(x, y) + d(z, w) = 8 > 4
$$

$$
= d(x, u) + d(y, u) + d(z, v) + d(w, v).
$$

If $u \in U$ and $v \in V$, or $u \in V$ and $v \in U$, then, respectively, for $x = a_1$, $y = a_2$, $z = b_1$, $w = b_2$, and for $x = b_1$, $y = b_2$, $z = a_1$, $w = a_2$,

$$
2d(u, v) + d(x, y) + d(z, w) \geq 6 > 4
$$

$$
= d(x, u) + d(y, u) + d(z, v) + d(w, v).
$$
Finally, if \( u \in N \) or \( v \in N \), then, respectively, for \( x = y = u \) and \( z, w \in N \) with \( d(z, w) = 2 \) and \( d(z, v) = d(w, v) = 1 \), and for \( z = w = v \) and \( x, y \in N \) with \( d(x, y) = 2 \) and \( d(x, u) = d(y, u) = 1 \),

\[
2d(u, v) + d(x, y) + d(z, w) \geq 4 > 2
= d(x, u) + d(y, u) + d(z, v) + d(w, v).
\]

The following example shows that the inequality (1.2) in the definition of the SLTP does not imply (1.1).

**Example 3.2.** Let \( M = \{a, b\} \cup \{u_i, v_i : i \in \mathbb{N}\} \) be a metric space where the distances between different points are defined as follows: for any \( i, j \in \mathbb{N} \), \( i \neq j \),

\[
d(a, b) = d(a, u_i) = d(b, u_i) = d(u_i, u_j) = d(v_i, v_j) = d(u_i, v_j) = 2
\]

and, for any \( i \in \mathbb{N} \),

\[
d(a, u_i) = d(b, v_i) = d(u_i, v_i) = 1.
\]

(See Fig. 2.) For any finite subset \( N \) of \( M \), we can find an \( i \in \mathbb{N} \) such that \( u_i, v_i \in M \setminus N \). We first show that, for any \( x, y, z, w \in N \),

\[
d(x, y) + d(z, w) + 2d(u_i, v_i) = d(x, y) + d(z, w) + 2
\leq d(x, u_i) + d(y, u_i) + d(z, v_i) + d(w, v_i).
\]

By symmetry it suffices to show that, for any \( x, y \in M \setminus \{u_i, v_i\} \),

\[
d(x, y) + 1 \leq d(x, u_i) + d(y, u_i).
\]

This inequality holds trivially if \( d(x, u_i) + d(y, u_i) \geq 3 \). It remains to note that if \( d(x, u_i) + d(y, u_i) < 3 \), then \( d(x, u_i) = d(y, u_i) = 1 \). Thus, \( x = y = a \), and the desired inequality trivially holds.

We now show that \( M \) does not have the LTP. Take \( N := \{a, b\} \). Then, for any \( u, v \in M \), \( u \neq v \), there exist \( x, y \in N \) such that

\[
d(x, y) + d(u, v) \geq d(x, u) + d(y, v) + 1.
\]

Indeed, set \( U := \{u_i : i \in \mathbb{N}\} \) and \( V := \{v_i : i \in \mathbb{N}\} \), and suppose that \( u, v \in M \), \( u \neq v \). If \( u, v \in U \) or \( u, v \in V \), then, for \( x = a, y = b \),

\[
d(x, y) + d(u, v) = 4 \geq 3 = d(x, u) + d(y, v).
\]
If \( u \in U \) and \( v \in V \), or \( u \in V \) and \( v \in U \), then, respectively, for \( x = a, y = b \), and for \( x = b, y = a \),

\[
d(x, y) + d(u, v) \geq 3 > 2 = d(x, u) + d(y, v).
\]

Finally, if \( u \in N \) or \( v \in N \), then, respectively, for \( x = u, y \in N \setminus \{x\} \), and for \( y = v, x \in N \setminus \{y\} \),

\[
d(x, y) + d(u, v) \geq 3 > 2 \geq d(x, u) + d(y, v).
\]

In [10, Proposition 4.7] it was shown that every infinite subset \( M \) of \( \ell_1 \), viewed as a metric space, has the LTP. It turns out that every such \( M \) has even the SLTP.

**Example 3.3.** Every infinite subset \( M \) of \( \ell_1 \), viewed as a metric space, has the SLTP.

Indeed, from [7, Theorem 5.6] combined with our Theorem 2.1 it follows that every unbounded metric space and every metric space \( M \) with the property that \( \inf \{d(x, y): x, y \in M, x \neq y\} = 0 \) has the SLTP (this can also, without too much effort, be verified directly). Thus it suffices to consider the case when \( M \) is a bounded and uniformly discrete subset of \( \ell_1 \). In this case there exist \( R, r > 0 \) such that for any \( x, y \in M \), \( x \neq y \),

\[
r < d(x, y) < R.
\]

Let \( N \) be a finite subset of \( M \) and let \( \varepsilon > 0 \). Choose \( \delta > 0 \) such that \( \varepsilon r \geq 6\delta \). Since \( N \) is finite, there exists an \( n \in \mathbb{N} \) such that for any \( x = (x_i) \in N \)

\[
\sum_{i > n} |x_i| \leq \delta.
\]

Since \( M \) is infinite and bounded, there exist \( u = (u_i), v = (v_i) \in M, u \neq v \), such that

\[
\sum_{i \leq n} |u_i - v_i| \leq \delta.
\]

For any \( x = (x_i), y = (y_i) \in N \) and \( a = (a_i), b = (b_i) \in \{u, v\} \),

\[
\sum_{i} |x_i - y_i| \leq \sum_{i \leq n} (|x_i - a_i| + |y_i - b_i| + |a_i - b_i|) + \sum_{i > n} |x_i - y_i| \\
\leq \sum_{i \leq n} (|x_i - a_i| + |y_i - b_i|) + 3\delta
\]

and

\[
\sum_{i} |u_i - v_i| \leq \sum_{i > n} |u_i - v_i - x_i + y_i| + \sum_{i > n} |x_i - y_i| + \sum_{i \leq n} |u_i - v_i| \\
\leq \sum_{i > n} (|x_i - u_i| + |y_i - v_i|) + 3\delta.
\]

Therefore, for any \( x = (x_i), y = (y_i), z = (z_i), w = (w_i) \in N \)

\[
(1 - \varepsilon)(d(x, y) + d(u, v)) \leq d(x, y) + d(u, v) - 6\delta
\]

\[
= \sum_{i} |x_i - y_i| + \sum_{i} |u_i - v_i| - 6\delta
\]
\[
\begin{align*}
&\leq \sum_{i \leq n} (|x_i - u_i| + |y_i - v_i|) + 3\delta \\
&\quad + \sum_{i > n} (|x_i - u_i| + |y_i - v_i|) + 3\delta - 6\delta \\
&= \sum_{i} (|x_i - u_i| + |y_i - v_i|) \\
&= d(x, u) + d(y, v)
\end{align*}
\]

and
\[
\begin{align*}
&\quad \leq 2d(u, v) + d(x, y) + d(z, w) - 12\delta \\
&= 2 \sum_{i} |u_i - v_i| + \sum_{i} |x_i - y_i| + \sum_{i} |z_i - w_i| - 12\delta \\
&\leq \sum_{i > n} (|x_i - u_i| + |z_i - v_i| + |y_i - u_i| + |w_i - v_i|) + 6\delta \\
&\quad + \sum_{i \leq n} (|x_i - u_i| + |y_i - u_i| + |z_i - v_i| + |w_i - v_i|) + 6\delta - 12\delta \\
&= \sum_{i} (|x_i - u_i| + |y_i - u_i| + |z_i - v_i| + |w_i - v_i|) \\
&= d(x, u) + d(y, u) + d(z, v) + d(w, v).
\end{align*}
\]

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References