Journal of Computational and Applied Mathematics 236 (2012) 3349–3359

Contents lists available at SciVerse ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Piecewise polynomial collocation for linear boundary value problems of fractional differential equations☆

Arvet Pedas, Enn Tamme*

Institute of Mathematics, University of Tartu, Liivi 2, 50409 Tartu, Estonia

ARTICLE INFO

Article history:
Received 1 December 2011
Received in revised form 22 February 2012

MSC:
65L60

Keywords:
Fractional differential equation
Caputo derivative
Boundary value problem
Spline collocation method
Graded grid

ABSTRACT

We consider a class of boundary value problems for linear multi-term fractional differential equations which involve Caputo-type fractional derivatives. Using an integral equation reformulation of the boundary value problem, some regularity properties of the exact solution are derived. Based on these properties, the numerical solution of boundary value problems by piecewise polynomial collocation methods is discussed. In particular, we study the attainable order of convergence of proposed algorithms and show how the convergence rate depends on the choice of the grid and collocation points. Theoretical results are verified by two numerical examples.

1. Introduction

In this paper we study the convergence behavior of a collocation method for the numerical solution of linear boundary value problems of the form

$$(D_*^{\alpha_p} y)(t) + \sum_{i=0}^{p-1} a_i(t)(D_*^{\alpha_i} y)(t) = f(t), \quad 0 \le t \le b, \tag{1.1}$$

$$\sum_{j=0}^{n_0} \alpha_{ij} y^{(j)}(0) + \sum_{j=0}^{n_1} \beta_{ij} y^{(j)}(b_1) = \gamma_i, \quad 0 < b_1 \le b,\ i = 0, \dots, n-1, \tag{1.2}$$

where

$$\begin{aligned} &0 \le \alpha_0 < \alpha_1 < \cdots < \alpha_p, \qquad n-1 < \alpha_p \le n, \quad n, p \in \mathbb{N} := \{1, 2, \dots\}, \\ &0 \le n_0 \le n-1, \qquad 0 \le n_1 \le n-1, \qquad \gamma_i, \alpha_{ij}, \beta_{ij} \in \mathbb{R} := (-\infty, \infty), \end{aligned} \tag{1.3}$$

a_i $(i = 0, \dots, p-1)$ and f are some given continuous functions from $[0, b]$ into $\mathbb{R}$. In (1.1) $D_*^0 := I$ is the identity operator and D_*^α is the Caputo differential operator of order $\alpha > 0$ defined by (see, e.g., [1])

$$(D_*^\alpha y)(t) := (D^\alpha (y - \mathcal{Q}_{k-1}[y]))(t), \quad k-1 < \alpha \le k,\ k \in \mathbb{N},\ t > 0.$$

☆ This work was supported by Estonian Science Foundation Grant No. 9104.
* Corresponding author. Tel.: +372 737 5863; fax: +372 737 5863.
E-mail addresses: Arvet.Pedas@ut.ee (A. Pedas), Enn.Tamme@ut.ee (E. Tamme).

0377-0427/$ – see front matter
doi:10.1016/j.cam.2012.03.002

Here

$$\mathcal{Q}_{k-1}[y](s) := \sum_{i=0}^{k-1} \frac{y^{(i)}(0)}{i!}\, s^i$$

is the Taylor polynomial of degree $k-1$ for y, centered at 0, and $D^\alpha y$ is the Riemann–Liouville fractional derivative of order α:

$$(D^\alpha y)(t) := (J^{k-\alpha} y)^{(k)}(t), \quad k-1 < \alpha \le k,\ k \in \mathbb{N},\ t > 0,$$

with $J^0 := I$ and J^α, the Riemann–Liouville integral operator, defined for $\alpha > 0$ by the formula

$$(J^\alpha y)(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\, y(s)\, ds, \quad t > 0,$$

where Γ is the Euler gamma function. If $\alpha = k \in \mathbb{N}$ then $D^k y = D_*^k y = y^{(k)}$ where $y^{(k)}$ is the usual k-th order derivative of y.

It is well known (see, e.g., [2,3]) that $J^\alpha,\ \alpha > 0$, is linear, bounded and compact as an operator from $L^\infty(0, b)$ into $C[0, b]$, and for any $y \in L^\infty(0, b)$

$$(J^\alpha y)^{(k)} \in C[0, b], \quad (J^\alpha y)^{(k)}(0) = 0,\ \alpha > 0,\ k = 0, \ldots, \lceil \alpha \rceil - 1, \tag{1.4}$$

$$J^\alpha J^\beta y = J^{\alpha+\beta} y, \quad \alpha > 0,\ \beta > 0, \tag{1.5}$$

$$D^\beta J^\alpha y = D_*^\beta J^\alpha y = J^{\alpha-\beta} y, \quad 0 < \beta \le \alpha, \tag{1.6}$$

where $\lceil \alpha \rceil$ is the smallest integer not less than α.

Fractional differential equations arise in various areas of science and engineering. In the last few decades the theory and numerical analysis of fractional differential equations have received increasing attention (see, for example, [1,3–5] and references cited in these books). Various existence and uniqueness results for boundary value problems of fractional differential equations are obtained in many recent publications (see [1,6–10] and the references therein).

A great deal of papers are devoted to the numerical solution of initial value problems for fractional differential equations (see, e.g., [1,11–16]). In contrast to this, only a few papers concern the numerical solution of boundary value problems for fractional differential equations. Numerical schemes based on a shooting algorithm are discussed in [17] and some algorithms based on the collocation method, piecewise polynomial collocation method and Haar wavelet method are proposed in [18–20], respectively.

In the present paper, the numerical solution of linear boundary value problems (1.1), (1.2) by piecewise polynomial collocation methods is under consideration. These methods have been shown to be efficient to solve integral equations, integro-differential equations (see, e.g., [2,21–25]) and fractional initial value problems (see [16,26]). Our aim is to present a complete analysis of the convergence of spline collocation solutions for problem (1.1), (1.2) in a situation where the derivatives of the functions $f(t)$ and $a_i(t)$ $(i = 0, \ldots, p-1)$ may be unbounded at $t = 0$. Our approach is based on some ideas and results of [16].

The remainder of the present paper is arranged as follows. In Section 2 we prove Theorem 2.1 which gives the estimates for higher order derivatives of the exact solution of problem (1.1), (1.2). These estimates will play a key role in the convergence analysis of proposed algorithms in Section 4. In Section 3 the description of a piecewise polynomial collocation method is given. We use an integral equation reformulation of the problem and special non-uniform grids reflecting the possible singular behavior of the exact solution. In Section 4 we prove the convergence of our method, derive global convergence estimates and analyze a (global) superconvergence effect for a special choice of collocation points. The main results of the paper are formulated in Theorems 4.1 and 4.2. Finally, in Section 5 the obtained theoretical results are verified by two numerical examples.

2. Smoothness of the solution

An important question that arises by studying the attainable order of the convergence of a numerical method is the question for the smoothness properties of the exact solution of a fractional differential equation. Some information about the smoothness properties of the solution can be obtained by using asymptotic expansions of the solution in fractional powers with respect to the independent variable (see, e.g., [1]).

In the present paper we use another approach: we introduce a weighted space of smooth functions $C^{q,\nu}(0, b]$ (cf., e.g., [2]) and show that the derivative $D_*^{\alpha_p} y$ of y, the solution of problem (1.1), (1.2), belongs to $C^{q,\nu}(0, b]$ (see Theorem 2.1). Here, by $C^{q,\nu}(0, b]$ $(q \in \mathbb{N},\ -\infty < \nu < 1)$ we denote the set of continuous functions $y : [0, b] \to \mathbb{R}$ which are q times continuously differentiable in $(0, b]$ and such that for all $t \in (0, b]$ and $i = 1, \ldots, q$ the following estimates hold:

$$|y^{(i)}(t)| \le c \begin{cases} 1 & \text{if } i < 1-\nu, \\ 1 + |\log t| & \text{if } i = 1-\nu, \\ t^{1-\nu-i} & \text{if } i > 1-\nu, \end{cases}$$

where $c = c(y)$ is a positive constant. Equipped with the norm

$$\|y\|_{q,\nu} := \max_{0\le t\le b} |y(t)| + \sum_{i=1}^{q} \sup_{0<t\le b} \left(w_{i+\nu-1}(t)\,|y^{(i)}(t)|\right),$$

$C^{q,\nu}(0, b]$ is a Banach space. Here

$$w_\lambda(t) := \begin{cases} 1 & \text{for } \lambda < 0, \\ (1 + |\log t|)^{-1} & \text{for } \lambda = 0, \\ t^\lambda & \text{for } \lambda > 0. \end{cases}$$

Clearly,

$$C^q[0, b] \subset C^{q,\nu}(0, b] \subset C^{m,\mu}(0, b] \subset C[0, b], \quad q \ge m \ge 1,\ \nu \le \mu < 1. \tag{2.1}$$

If $a, v \in C^{q,\nu}(0, b],\ q \in \mathbb{N},\ \nu < 1$, then (see [2]) $av \in C^{q,\nu}(0, b]$ and

$$\|av\|_{q,\nu} \le c\,\|a\|_{q,\nu}\|v\|_{q,\nu}, \tag{2.2}$$

with a positive constant c which is independent of a and v. Note that a function of the form $y(t) = g_1(t)\,t^\mu + g_2(t)$ is included in $C^{q,\nu}(0, b]$ if $\mu \ge 1 - \nu > 0$ and $g_j \in C^q[0, b],\ j = 1, 2$.

In what follows we use an integral equation reformulation of the problem (1.1), (1.2) introducing a new unknown function $z := D_*^{\alpha_p} y$. To reach the desirable reformulation (see (2.9)) let us consider an equation

$$D_*^{\alpha_p} y = z, \quad n - 1 < \alpha_p \le n,\ n \in \mathbb{N}, \tag{2.3}$$

where z is an arbitrary function in $C[0, b]$. The solutions of Eq. (2.3) have the following form (see [1,3]):

$$y(t) = (J^{\alpha_p} z)(t) + \sum_{k=0}^{n-1} c_k\, t^k \tag{2.4}$$

where $c_k \in \mathbb{R}$ $(k = 0, \dots, n - 1)$ are arbitrary constants. The function (2.4) satisfies the boundary conditions (1.2) if and only if (see (1.4) and (1.6))

$$\sum_{j=0}^{n_0} \alpha_{ij}\, j!\, c_j + \sum_{j=0}^{n_1} \beta_{ij}\left[(J^{\alpha_p - j} z)(b_1) + \sum_{k=j}^{n-1} \frac{k!}{(k-j)!}\, b_1^{k-j}\, c_k\right] = \gamma_i, \quad i = 0, \dots, n - 1.$$

We rewrite these equations in the form

$$\sum_{j=0}^{n-1}\left[j!\,\alpha_{ij} + \sum_{k=0}^{j} \beta_{ik} \frac{j!}{(j-k)!} b_1^{j-k}\right] c_j = \gamma_i - \sum_{j=0}^{n_1} \beta_{ij} (J^{\alpha_p - j} z)(b_1), \quad i = 0, \dots, n - 1, \tag{2.5}$$

setting $\alpha_{ij} = 0$ for $j > n_0$ and $\beta_{ij} = 0$ for $j > n_1$. Clearly, (2.5) is a linear system of equations with respect to $c_0, \dots, c_{n-1}$. In the sequel we assume that the matrix M of the system (2.5) is regular. Observe that M is regular if and only if from all polynomials y of degree $n - 1$ only $y = 0$ satisfies the conditions (1.2) with $\gamma_i = 0,\ i = 0, \dots, n - 1$.

Let $M^{-1} = (p_{ij})_{i,j=0}^{n-1}$ be the inverse of M. Using M^{-1} the solution of the system (2.5) can be written in the form

$$c_k = d_k - \sum_{j=0}^{n_1} \delta_{kj} (J^{\alpha_p - j} z)(b_1), \quad k = 0, \dots, n - 1,$$

where

$$d_k := \sum_{l=0}^{n-1} p_{kl}\, \gamma_l, \qquad \delta_{kj} := \sum_{l=0}^{n-1} p_{kl}\, \beta_{lj}.$$

Thus, a solution y of Eq. (2.3) in the form

$$y = Gz + Q \tag{2.6}$$

satisfies the conditions (1.2) if and only if

$$(Gz)(t) := (J^{\alpha_p} z)(t) - \sum_{k=0}^{n-1} t^k \sum_{j=0}^{n_1} \delta_{kj} (J^{\alpha_p - j} z)(b_1), \quad 0 \le t \le b, \tag{2.7}$$

$$Q(t) := \sum_{k=0}^{n-1} d_k\, t^k. \tag{2.8}$$

Suppose now that y is a solution of the problem (1.1), (1.2) such that $D_*^{\alpha_p} y \in C[0,b]$. Then it follows from the observations above that y has the form (2.6) where $z = D_*^{\alpha_p} y \in C[0,b]$ and G and Q are given by the formulas (2.7) and (2.8), respectively. Inserting (2.6) into (1.1), we see that $z = D_*^{\alpha_p} y$ satisfies an equation of the form

$$z = Tz + g \tag{2.9}$$

where

$$Tz := -\sum_{i=0}^{p-1} a_i D_*^{\alpha_i} Gz, \qquad g := f - \sum_{i=0}^{p-1} a_i D_*^{\alpha_i} Q. \tag{2.10}$$

Conversely, it is easy to show that if $z \in C[0,b]$ is a solution of Eq. (2.9) then y defined by (2.6) is a solution of the problem (1.1), (1.2). In this sense Eq. (2.9) is equivalent to the boundary value problem (1.1), (1.2) and we can use it by constructing of high order methods for the numerical solution of (1.1), (1.2).

Note that $D_*^{\alpha_i} Gz$ and $D_*^{\alpha_i} Q$ $(i = 0, \ldots, p-1)$ in (2.10) can be found in the following way. Let $v_k(t) := t^k,\ t > 0,\ k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Then (see, e.g., [1])

$$(D_*^{\alpha_i} v_k)(t) = \begin{cases} 0 & \text{if } k = 0, \ldots, \lceil \alpha_i \rceil - 1, \\ \dfrac{k!}{\Gamma(1+k-\alpha_i)}\, t^{k-\alpha_i} & \text{if } k \geq \lceil \alpha_i \rceil. \end{cases}$$

This together with (1.6), (2.7) and (2.8) yields that

$$(D_*^{\alpha_i} Gz)(t) = \begin{cases} (J^{\alpha_p - \alpha_i} z)(t) - \displaystyle\sum_{k=\lceil \alpha_i \rceil}^{n-1} \frac{k!}{\Gamma(1+k-\alpha_i)}\, t^{k-\alpha_i} \sum_{j=0}^{n_1} \delta_{kj} (J^{\alpha_p - j} z)(b_1) & \text{if } 0 \leq \alpha_i \leq n-1, \\ (J^{\alpha_p - \alpha_i} z)(t) & \text{if } n-1 < \alpha_i < n, \end{cases} \tag{2.11}$$

$$(D_*^{\alpha_i} Q)(t) = \begin{cases} \displaystyle\sum_{k=\lceil \alpha_i \rceil}^{n-1} \frac{k!}{\Gamma(1+k-\alpha_i)}\, d_k t^{k-\alpha_i} & \text{if } 0 \leq \alpha_i \leq n-1, \\ 0 & \text{if } n-1 < \alpha_i < n, \end{cases} \tag{2.12}$$

where $t \in [0,b]$. In (2.11) we have

$$(J^{\alpha_p - \alpha_i} z)(t) = \frac{1}{\Gamma(\alpha_p - \alpha_i)} \int_0^t (t-s)^{\alpha_p - \alpha_i - 1} z(s)\, ds, \quad i = 0, \ldots, p-1, \tag{2.13}$$

$$(J^{\alpha_p - j} z)(b_1) = \frac{1}{\Gamma(\alpha_p - j)} \int_0^{b_1} (b_1 - s)^{\alpha_p - j - 1} z(s)\, ds, \quad j = 0, \ldots, n_1. \tag{2.14}$$

On these observations we find that Eq. (2.9) is a linear Fredholm type integral equation of the second kind. Moreover, it follows from (2.10)–(2.14) that the kernel of this equation may be weakly singular at $s = t$.

Theorem 2.1. *Let (1.3) be true and assume that* $a_i \in C^{q,\mu}(0,b]$ $(i = 0, \ldots, p-1)$ *and* $f \in C^{q,\mu}(0,b]$, *where* $q \in \mathbb{N}$ *and* $-\infty < \mu < 1$. *Moreover, assume that the problem (1.1), (1.2) with* $f = 0$ *and* $\gamma_i = 0$ $(i = 0, \ldots, n-1)$ *has in* $C[0,b]$ *only the trivial solution* $y = 0$, *and from all polynomials* y *of degree* $n-1$ *only* $y = 0$ *satisfies the conditions (1.2) with* $\gamma_i = 0,\ i = 0, \ldots, n-1$.

Then boundary value problem (1.1), (1.2) possesses a unique solution $y \in C^{n-1}[0,b]$ *such that* $D_*^{\alpha_p} y \in C[0,b]$. *For this solution there holds* $D_*^{\alpha_p} y \in C^{q,\nu}(0,b]$ *where*

$$\nu := \max\{\mu, \nu_1, \nu_2\} \tag{2.15}$$

with

$$\nu_1 := \max\{1 - \alpha_p + \alpha_i : \alpha_p - \alpha_i \notin \mathbb{N},\ i = 0, \ldots, p-1\},$$
$$\nu_2 := \max\{1 - \lceil \alpha_i \rceil + \alpha_i : \alpha_i < n-1,\ \alpha_i \notin \mathbb{N}_0,\ i = 0, \ldots, p-1\}.$$

If for all indices $i = 0, \ldots, p-1$ *we have* $\alpha_p - \alpha_i \in \mathbb{N}$ *then we may set* ν_1 *to be equal to any number which is less than* 1. *Analogously, if we have* $\alpha_i \in \mathbb{N}_0$ *for all indices* $i = 0, \ldots, p-1$ *such that* $\alpha_i < n-1$ *then we may set* ν_2 *to be equal to any number less than* 1.

Proof. Let us consider Eq. (2.9) which is equivalent to the problem (1.1), (1.2). From (2.12) it follows that if $\alpha_i \in \mathbb{N}_0$ or $\alpha_i > n-1$ then (see (2.1)) $D_*^{\alpha_i} Q \in C^q[0,b] \subset C^{q,\nu}(0,b]$ with arbitrary $q \in \mathbb{N}$ and $\nu < 1$. If $\alpha_i \notin \mathbb{N}_0$ and $0 < \alpha_i < n-1$ then $D_*^{\alpha_i} Q \in C^{q,1-\lceil \alpha_i \rceil + \alpha_i}(0,b] \subset C^{q,\nu_2}(0,b]$. This together with (2.2) and (2.10) yields that $g \in C^{q,\tilde{\nu}}(0,b] \subset C^{q,\nu}(0,b]$ with $\tilde{\nu} := \max\{\mu, \nu_2\}$ and ν defined by (2.15). Thus, the forcing function g in Eq. (2.9) belongs to $C^{q,\nu}(0,b]$.

Further, it follows from [2] that $J^{\alpha_p-\alpha_i}$ $(i = 0, \dots, p-1)$ is linear and compact as operator from $C^{q,\nu}(0, b]$ into $C^{q,\nu}(0, b]$. Linear functionals $J_1^{\alpha_p-j} : C^{q,\nu}(0, b] \to \mathbb{R}$, $j = 0, \dots, n_1$, defined by $J_1^{\alpha_p-j}z := (J^{\alpha_p-j}z)(b_1)$ are bounded and consequently compact in $C^{q,\nu}(0, b]$. Using (2.2) we obtain that T defined by (2.10) is linear and compact as an operator from $C^{q,\nu}(0, b]$ into $C^{q,\nu}(0, b]$. Since the homogeneous equation $z = Tz$ has in $C^{q,\nu}(0, b] \subset C[0, b]$ only the trivial solution $z = 0$, it follows from the Fredholm alternative theorem that Eq. (2.9) has a unique solution $z \in C^{q,\nu}(0, b]$. Consequently, the problem (1.1), (1.2) possesses a unique solution $y = Gz + Q \in C^{n-1}[0, b]$ such that $D_*^{\alpha_p}y = z \in C^{q,\nu}(0, b] \subset C[0, b]$. □

Remark 2.1. If in Eq. (1.1) $a_i \in C^q[0, b]$ $\quad (i = 0, \dots, p-1)$ and $f \in C^q[0, b]$ where $q \in \mathbb{N}$ then in Theorem 2.1 we may set $\nu := \max\{\nu_1, \nu_2\}$.

Remark 2.2. If Eq. (1.1) has the form

$$D_*^{\alpha_1}y(t) + a_0(t)y(t) = f(t), \quad 0 \le t \le b,\ n-1 < \alpha_1 < n,\ n \in \mathbb{N}, \tag{2.16}$$

then the boundary value problem (2.16), (1.2) is equivalent to Eq. (2.9) with $Tz = -a_0Gz$ and $g = f - a_0Q$. Under the assumptions of Theorem 2.1 with $p = 1$ we now obtain that the problem (2.16), (1.2) possesses a unique solution y such that $D_*^{\alpha_1}y \in C^{q,\nu}(0, b]$ with $\nu := \max\{\mu, 1 - \alpha_1\}$.

Remark 2.3. We already mentioned above that Eq. (2.9) can be considered as a Fredholm integral equation with a kernel $K(t, s)$ which may have a weak singularity at $s = t$. Let $[0, b_1]$ (see (1.2)) be the interval of integration in (2.9) and let $z(t)$ be the solution to (2.9) for $0 \le t \le b_1$. In this case the derivatives of $z(t)$ as the derivatives of the solution of a weakly singular Fredholm integral equation might be singular at the endpoints $t = 0$ and $t = b_1$ of the interval $[0, b_1]$ (see, e.g., [21]). However, it follows from Theorem 2.1 that the derivatives of $z(t)$ have no singularities at $t = b_1$ and, if $b > b_1$, then $z(t)$ has a smooth extension from $[0, b_1]$ to $[0, b]$. A numerical confirmation to this effect (to Theorem 2.1) will be given by Example 5.1 (see also [17]).

3. Spline collocation method

Let $N \in \mathbb{N}$ and let $\Pi_N := \{t_0, \dots, t_N\}$ be a partition (a graded grid) of the interval $[0, b]$ with the grid points

$$t_j := b\left(\frac{j}{N}\right)^r, \quad j = 0, 1, \dots, N, \tag{3.1}$$

where the grading exponent $r \in \mathbb{R}$, $r \ge 1$. If $r = 1$, then the grid points (3.1) are distributed uniformly; for $r > 1$ the points (3.1) are more densely clustered near the left endpoint of the interval $[0, b]$.

For given integer $k \ge 0$ by $S_k^{(-1)}(\Pi_N)$ is denoted the standard space of piecewise polynomial functions:

$$S_k^{(-1)}(\Pi_N) := \left\{v : v|_{(t_{j-1},t_j)} \in \pi_k,\ j = 1, \dots, N\right\}.$$

Here $v|_{(t_{j-1},t_j)}$ is the restriction of $v : [0, b] \to \mathbb{R}$ onto the subinterval (t_{j-1}, t_j) and π_k denotes the set of polynomials of degree not exceeding k. Note that the elements of $S_k^{(-1)}(\Pi_N)$ may have jump discontinuities at the interior points $t_1, \dots, t_{N-1}$ of the grid Π_N.

In every interval $[t_{j-1}, t_j]$, $j = 1, \dots, N$, we define $m \in \mathbb{N}$ collocation points $t_{j1}, \dots, t_{jm}$ by formula

$$t_{jk} := t_{j-1} + \eta_k(t_j - t_{j-1}), \quad k = 1, \dots, m,\ j = 1, \dots, N, \tag{3.2}$$

where $\eta_1 \dots, \eta_m$ are some fixed (collocation) parameters which do not depend on j and N and satisfy

$$0 \le \eta_1 < \eta_2 < \cdots < \eta_m \le 1. \tag{3.3}$$

We look for an approximate solution y_N of the boundary value problem (1.1), (1.2) in the form

$$y_N = Gz_N + Q \tag{3.4}$$

where G and Q are defined by (2.7) and (2.8), respectively, and $z_N \in S_{m-1}^{(-1)}(\Pi_N)(m, N \in \mathbb{N})$ is determined by the following collocation conditions:

$$z_N(t_{jk}) = (Tz_N)(t_{jk}) + g(t_{jk}), \quad k = 1, \dots, m,\ j = 1, \dots, N. \tag{3.5}$$

Here g, T and t_{jk} are defined by (2.10) and (3.2), respectively. If $\eta_1 = 0$, then by $z_N(t_{j1})$ we denote the right limit $\lim_{t\to t_{j-1},t>t_{j-1}} z_N(t)$. If $\eta_m = 1$, then $z_N(t_{jm})$ denotes the left limit $\lim_{t\to t_j,t<t_j} z_N(t)$. Conditions (3.5) have an operator equation representation

$$z_N = \mathcal{P}_N Tz_N + \mathcal{P}_N g \tag{3.6}$$

with an interpolation operator $\mathcal{P}_N = \mathcal{P}_{N,m} : C[0, b] \to S_{m-1}^{(-1)}(\Pi_N)$ defined for any $v \in C[0, b]$ by the following conditions:

$$\mathcal{P}_N v \in S_{m-1}^{(-1)}(\Pi_N), \qquad (\mathcal{P}_N v)(t_{jk}) = v(t_{jk}), \quad k = 1, \dots, m,\ j = 1, \dots, N. \tag{3.7}$$

The collocation conditions (3.5) form a system of equations whose exact form is determined by the choice of a basis in $S_{m-1}^{(-1)}(\Pi_N)$. If $\eta_1 > 0$ or $\eta_m < 1$ then we can use the Lagrange fundamental polynomial representation:

$$z_N(t) = \sum_{\lambda=1}^{N}\sum_{\mu=1}^{m} c_{\lambda\mu}\varphi_{\lambda\mu}(t), \quad t \in [0, b], \tag{3.8}$$

where $\varphi_{\lambda\mu}(t) := 0$ for $t \notin [t_{\lambda-1}, t_\lambda]$ and

$$\varphi_{\lambda\mu}(t) := \prod_{i=1, i\neq\mu}^{m} \frac{t - t_{\lambda i}}{t_{\lambda\mu} - t_{\lambda i}} \quad \text{for } t \in [t_{\lambda-1}, t_\lambda],\ \mu = 1, \ldots, m,\ \lambda = 1, \ldots, N.$$

Then $z_N \in S_{m-1}^{(-1)}(\Pi_N)$ and $z_N(t_{jk}) = c_{jk},\ k = 1, \ldots, m,\ j = 1, \ldots, N$. Searching the solution of (3.5) in the form (3.8), we obtain a system of linear algebraic equations with respect to the coefficients $c_{jk} = z_N(t_{jk})$:

$$c_{jk} = \sum_{\lambda=1}^{N}\sum_{\mu=1}^{m} (T\varphi_{\lambda\mu})(t_{jk})c_{\lambda\mu} + g(t_{jk}), \quad k = 1, \ldots, m,\ j = 1, \ldots, N. \tag{3.9}$$

Note that this algorithm can be used also in the case if in (3.3) $\eta_1 = 0$ and $\eta_m = 1$. In this case we have $t_{jm} = t_{j+1,1} = t_j,\ c_{jm} = c_{j+1,1} = z_N(t_j)$ $(j = 1, \ldots, N-1)$, and hence in the system (3.9) there are $(m-1)N + 1$ equations and unknowns.

4. Convergence analysis

In order to investigate the convergence of our method we need the following result from [16].

Lemma 4.1. *Let $z \in C^{q,\nu}(0, b]$, where $-\infty < \nu < 1$ and $q := m + \min\{m, \lceil\alpha\rceil\}$, with some $m \in \mathbb{N}$ and $\alpha \in \mathbb{R},\ \alpha \geq 0$. Moreover, assume that a quadrature approximation*

$$\int_0^1 F(x)\,dx \approx \sum_{k=1}^{m} w_k F(\eta_k) \tag{4.1}$$

with the knots $\{\eta_k\}$ satisfying (3.3) and appropriate weights $\{w_k\}$ is exact for all polynomials of degree $q - 1$. Then, for all values of the grid parameter $r \in [1, \infty)$ in (3.1), we have

$$\|J^\alpha(z - \mathcal{P}_N z)\|_\infty \leq c \begin{cases} E_N(m, \alpha, \nu, r) & \text{if } 0 \leq \alpha < 1, \\ \Theta_N(m + \alpha, \nu, r) & \text{if } 1 \leq \alpha \leq m, \\ \Theta_N(2m, \nu, r) & \text{if } \alpha \geq m. \end{cases} \tag{4.2}$$

Here c is a constant which is independent of N, $\mathcal{P}_N$ is given by (3.7) and

$$E_N(m, \alpha, \nu, r) := \begin{cases} N^{-r(1+\alpha-\nu)} & \text{for } 1 \leq r < \dfrac{m+\alpha}{1+\alpha-\nu}, \\ N^{-m-\alpha}(1 + \log N) & \text{for } r = \dfrac{m+\alpha}{1+\alpha-\nu} = 1, \\ N^{-m-\alpha} & \text{for } r = \dfrac{m+\alpha}{1+\alpha-\nu} > 1 \\ & \text{or } r > \dfrac{m+\alpha}{1+\alpha-\nu}, \end{cases} \tag{4.3}$$

$$\Theta_N(q, \nu, r) := \begin{cases} N^{-r(2-\nu)} & \text{for } 1 \leq r < \dfrac{q}{2-\nu}, \\ N^{-q}(1 + \log N) & \text{for } r = \dfrac{q}{2-\nu} \geq 1, \\ N^{-q} & \text{for } r > \dfrac{q}{2-\nu}, \end{cases} \tag{4.4}$$

$$\|v\|_\infty := \sup_{0<t<b} |v(t)|, \quad v \in L^\infty(0, b).$$

Remark 4.1. In the case $\alpha = 0$ the estimate (4.2) coincides with the corresponding result from [2]. Note that in this case we can choose the weights $w_1, \ldots, w_m$ so that the quadrature approximation (4.1) is exact for all polynomials of degree $m - 1$ for arbitrary parameters $\eta_1, \ldots, \eta_m$ satisfying (3.3).

Armed with Lemma 4.1 we can prove the convergence of our method and study the attainable order of convergence for arbitrary collocation parameters $\eta_1, \dots, \eta_m$ satisfying (3.3).

Theorem 4.1. *Let $m \in \mathbb{N}$ and assume that the collocation points (3.2) with grid points (3.1) and arbitrary parameters $\eta_1, \dots, \eta_m$ satisfying (3.3) are used. Let (1.3) be true and assume that $a_i \in C[0, b]$ $(i = 0, \dots, p-1)$ and $f \in C[0, b]$. Moreover, assume that the problem (1.1), (1.2) with $f = 0$ and $\gamma_i = 0$ $(i = 0, \dots, n-1)$ has in $C[0, b]$ only the trivial solution $y = 0$, and from all polynomials y of degree $n-1$ only $y = 0$ satisfies the conditions (1.2) with $\gamma_i = 0,\ i = 0, \dots, n-1$.*

Then problem (1.1), (1.2) has a unique solution $y \in C^{n-1}[0, b]$ such that $D_*^{\alpha_p} y \in C[0, b]$. Moreover, there exists an integer N_0 such that for all $N \geq N_0$ Eq. (3.6) possesses a unique solution $z_N \in S_{m-1}^{(-1)}(\Pi_N)$ and

$$\|y - y_N\|_\infty \to 0 \quad \text{as } N \to \infty \tag{4.5}$$

where y_N is defined by (3.4).

If, in addition, $a_i \in C^{m,\mu}(0, b]$, $(i = 0, \dots, p-1)$ and $f \in C^{m,\mu}(0, b]$ with $\mu < 1$, then for all $N \geq N_0$ and $r \geq 1$ the following error estimate holds:

$$\|y - y_N\|_\infty \leq c\, E_N(m, 0, \nu, r). \tag{4.6}$$

Here c is a constant which is independent of N and E_N and ν are defined by (2.15) and (4.3), respectively.

Proof. Since T defined by (2.10) is linear and compact as an operator from $L^\infty(0, b)$ into $C[0, b]$ (see the proof of Theorem 2.1) and Eq. $z = Tz$ has in $C[0, b]$ only the trivial solution $z = 0$, Eq. (2.9) has a unique solution $z \in C[0, b]$. Consequently, the problem (1.1), (1.2) possesses a unique solution $y = Gz + Q \in C^{n-1}[0, b]$ such that $D_*^{\alpha_p} y = z \in C[0, b]$. Using a standard argument (see, e.g., [2,26]), we obtain that there exists an integer N_0 such that for $N \geq N_0$ the operators $I - \mathcal{P}_N T$ are invertible in $L^\infty(0, b)$, Eq. (3.6) possesses a unique solution $z_N \in S_{m-1}^{(-1)}(\Pi_N)$ and

$$\|(I - \mathcal{P}_N T)^{-1}\|_{\mathcal{L}(L^\infty(0,b), L^\infty(0,b))} \leq c, \quad N \geq N_0. \tag{4.7}$$

Here the constant c does not depend on N and by $\mathcal{L}(L^\infty(0, b), L^\infty(0, b))$ is denoted the Banach space of bounded linear operators from $L^\infty(0, b)$ into $L^\infty(0, b)$. From (2.9) and (3.6) we get that

$$(I - \mathcal{P}_N T)(z - z_N) = z - \mathcal{P}_N z, \quad N \geq N_0,$$

and consequently,

$$\|z - z_N\|_\infty \leq c\, \|z - \mathcal{P}_N z\|_\infty, \quad N \geq N_0, \tag{4.8}$$

where c does not depend on N. As $J^{\alpha_p - j} \in \mathcal{L}(L^\infty(0, b), C[0, b])$, $j = 0, \dots, n_1$, then, due to (2.6), (3.4), (2.7) and (4.8),

$$\|y - y_N\|_\infty = \|G(z - z_N)\|_\infty \leq c\|z - z_N\|_\infty \leq c_1\, \|z - \mathcal{P}_N z\|_\infty, \quad N \geq N_0, \tag{4.9}$$

where c and c_1 are some constants not depending on N. Since $\|z - \mathcal{P}_N z\|_\infty \to 0$ for every $z \in C[0, b]$ as $N \to \infty$ (see [2]), we have justified the convergence (4.5).

If $a_i \in C^{m,\mu}(0, b]$ $(i = 0, \dots, p-1)$ and $f \in C^{m,\mu}(0, b]$ then from Theorem 2.1 it follows that $z \in C^{m,\nu}(0, b]$. Taking into account Remark 4.1, the estimate (4.6) follows from (4.2) and (4.9) with $\alpha = 0$. □

Remark 4.2. Theorems 2.1 and 4.1 give a basis for showing the well-posedness of fractional boundary value problems (1.1), (1.2) and for studying the numerical stability of proposed algorithms. Traditionally, a problem is called well-posed, if the following three conditions for this problem are fulfilled: (1) a solution exists, (2) the solution is unique and (3) the solution depends on the given data in a continuous way. The first two aspects have already been discussed in Section 2 (see Theorem 2.1); the third one requires further attention. Here some instructions can be found from [1]. Finally, in studying the numerical stability of the method (3.4), (3.6), the uniform boundedness of the norms of operators $(I - \mathcal{P}_N T)^{-1}$ will play a key role, see (4.7).

It follows from Theorem 4.1 that in the case of sufficiently smooth a_i $(i = 0, \dots, p-1)$ and f, using sufficiently large values of the grid parameter r, for method (3.4), (3.6) by every choice of collocation parameters $0 \leq \eta_1 < \cdots < \eta_m \leq 1$ a convergence of order $O(N^{-m})$ can be expected. In the following we show that by a careful choice of parameters $\eta_1, \dots, \eta_m$ it is possible to establish a faster convergence of this method.

Theorem 4.2. *Let the following conditions be fulfilled:*

(i) *$\mathcal{P}_N = \mathcal{P}_{N,m}$ $(N, m \in \mathbb{N})$ is defined by (3.7) where the interpolation nodes (3.2) with grid points (3.1) and parameters (3.3) are used;*
(ii) *the problem (1.1), (1.2) satisfies the assumptions of Theorem 2.1 with $q := m + \min\{m, \lceil \alpha_p - \beta \rceil\}$ where $\beta := \max\{\alpha_{p-1}, n_1\}$;*
(iii) *the quadrature approximation (4.1) is exact for all polynomials of degree $q - 1$.*

Then problem (1.1), (1.2) has a unique solution $y \in C^{n-1}[0,b]$ with $D_*^{\alpha_p} y \in C^{q,\nu}(0,b]$, there exists an integer N_0 such that, for $N \geq N_0$, Eq. (3.6) possesses a unique solution $z_N \in S_{m-1}^{(-1)}(\Pi_N)$ and the following error estimate holds:

$$\|y - y_N\|_\infty \leq c \begin{cases} E_N(m, \alpha_p - \beta, \nu, r) & \text{for } 0 < \alpha_p - \beta < 1, \\ \Theta_N(m + \alpha_p - \beta, \nu, r) & \text{for } 1 \leq \alpha_p - \beta \leq m, \\ \Theta_N(2m, \nu, r) & \text{for } \alpha_p - \beta \geq m. \end{cases} \tag{4.10}$$

Here c is a positive constant not depending on N, $r \in [1, \infty)$ is the grading exponent of the grid (see (3.1)) and ν, y_N, E_N and Θ_N are defined by (2.15), (3.4), (4.3) and (4.4), respectively.

Proof. Due to Theorem 2.1 Eq. (2.9) has a unique solution $z = D_*^{\alpha_p} y \in C^{q,\nu}(0,b] \subset C[0,b]$ and problem (1.1), (1.2) possesses a unique solution $y = Gz + Q \in C^{n-1}[0,b]$ with G and Q defined by (2.7) and (2.8), respectively. It follows from Theorem 4.1 that there exists an integer N_0 such that for $N \geq N_0$ Eq. (3.6) has a unique solution $z_N \in S_{m-1}^{(-1)}(\Pi_N)$. Let

$$\hat{z}_N := Tz_N + g, \quad N \geq N_0, \tag{4.11}$$

where T and g are defined by (2.10). From (3.6) and (4.11) we obtain that $\mathcal{P}_N \hat{z}_N = z_N$ and therefore

$$\hat{z}_N(t_{jk}) = z_N(t_{jk}), \quad k = 1, \ldots, m, \ j = 1, \ldots, N.$$

Substituting $z_N = \mathcal{P}_N \hat{z}_N$ into (4.11) we see that $\hat{z}_N$ is a solution of the equation

$$\hat{z}_N = T\mathcal{P}_N \hat{z}_N + g, \quad N \geq N_0. \tag{4.12}$$

From (2.9) and (4.12) it follows the identity

$$(I - T\mathcal{P}_N)(z - \hat{z}_N) = T(z - \mathcal{P}_N z), \quad N \geq N_0. \tag{4.13}$$

Since

$$(I - T\mathcal{P}_N)^{-1} = I + T(I - \mathcal{P}_N T)^{-1} \mathcal{P}_N, \quad N \geq N_0,$$

and $\|\mathcal{P}_N\|_{\mathcal{L}(C[0,b], L^\infty(0,b))} \leq c$ (see [2]), we get with help of (4.13), (4.7), (2.10) and (2.11) for $N \geq N_0$ that

$$\begin{aligned} \|z - \hat{z}_N\|_\infty &\leq c \, \|T(z - \mathcal{P}_N z)\|_\infty \\ &\leq c_1 \sum_{i=0}^{p-1} \|J^{\alpha_p - \alpha_i}(z - \mathcal{P}_N z)\|_\infty + c_2 \sum_{j=0}^{n_1} |J^{\alpha_p - j}(z - \mathcal{P}_N z)(b_1)|, \end{aligned}$$

with some constants c, c_1 and c_2 which do not depend on N. Using (1.5) and the boundedness of J^α, $\alpha > 0$, we obtain for $N \geq N_0$ the following estimates:

$$\begin{aligned} &\|J^{\alpha_p - \alpha_i}(z - \mathcal{P}_N z)\|_\infty \leq c \, \|J^{\alpha_p - \alpha_{p-1}}(z - \mathcal{P}_N z)\|_\infty, \quad i = 0, \ldots, p-1, \\ &|J^{\alpha_p - j}(z - \mathcal{P}_N z)(b_1)| \leq \|J^{\alpha_p - j}(z - \mathcal{P}_N z)\|_\infty \\ &\qquad \leq c \, \|J^{\alpha_p - n_1}(z - \mathcal{P}_N z)\|_\infty, \quad j = 0, \ldots, n_1, \end{aligned}$$

with c not depending on N. Therefore (see (4.2)),

$$\begin{aligned} \|z - \hat{z}_N\|_\infty &\leq c \, \|J^{\alpha_p - \beta}(z - \mathcal{P}_N z)\|_\infty \\ &\leq c_1 \begin{cases} E_N(m, \alpha_p - \beta, \nu, r) & \text{if } 0 < \alpha_p - \beta < 1, \\ \Theta_N(m + \alpha_p - \beta, \nu, r) & \text{if } 1 \leq \alpha_p - \beta \leq m, \\ \Theta_N(2m, \nu, r) & \text{if } \alpha_p - \beta \geq m \end{cases} \end{aligned} \tag{4.14}$$

where $\beta = \max\{\alpha_{p-1}, n_1\}$ and the constants c and c_1 do not depend on N. Further, for $N \geq N_0$ we have $z - z_N = (z - \mathcal{P}_N z) + \mathcal{P}_N(z - \hat{z}_N)$ and thus

$$\|y - y_N\|_\infty = \|G(z - z_N)\|_\infty \leq \|G(z - \mathcal{P}_N z)\|_\infty + c \, \|z - \hat{z}_N\|_\infty,$$

where the constant c does not depend on N. This together with (2.7), (4.2) and (4.14) yields the estimate (4.10). □

5. Numerical examples

In this section, we present some numerical experiments to demonstrate the accuracy of the spline collocation method (3.4), (3.6) and compare the actual convergence rate with the theoretical estimates (4.10) and (4.14).

Table 5.1
Results for the problem (5.1).

N	$r = 1$		$r = 2$		$r = 2.5$	
	ε_N	ϱ_N	ε_N	ϱ_N	ε_N	ϱ_N
4	$3.62 \cdot 10^{-2}$	1.44	$1.45 \cdot 10^{-2}$	4.65	$3.17 \cdot 10^{-2}$	4.71
8	$1.96 \cdot 10^{-2}$	1.85	$3.62 \cdot 10^{-3}$	3.99	$5.67 \cdot 10^{-3}$	5.59
16	$1.01 \cdot 10^{-2}$	1.94	$1.03 \cdot 10^{-3}$	3.51	$9.53 \cdot 10^{-4}$	5.95
32	$5.14 \cdot 10^{-3}$	1.97	$2.80 \cdot 10^{-4}$	3.68	$1.63 \cdot 10^{-4}$	5.83
64	$2.59 \cdot 10^{-3}$	1.98	$7.38 \cdot 10^{-5}$	3.80	$2.94 \cdot 10^{-5}$	5.56
128	$1.30 \cdot 10^{-3}$	1.98	$1.91 \cdot 10^{-5}$	3.87	$5.23 \cdot 10^{-6}$	5.62
256	$6.52 \cdot 10^{-4}$	2.00	$4.88 \cdot 10^{-6}$	3.91	$9.25 \cdot 10^{-7}$	5.66
N	$\hat{\varepsilon}_N$	$\hat{\varrho}_N$	$\hat{\varepsilon}_N$	$\hat{\varrho}_N$	$\hat{\varepsilon}_N$	$\hat{\varrho}_N$
4	$6.81 \cdot 10^{-3}$	3.47	$9.49 \cdot 10^{-3}$	3.84	$1.67 \cdot 10^{-2}$	3.66
8	$2.78 \cdot 10^{-3}$	2.45	$1.87 \cdot 10^{-3}$	5.07	$3.64 \cdot 10^{-3}$	4.58
16	$1.04 \cdot 10^{-3}$	2.67	$3.47 \cdot 10^{-4}$	5.40	$7.39 \cdot 10^{-4}$	4.93
32	$3.78 \cdot 10^{-4}$	2.75	$6.32 \cdot 10^{-5}$	5.48	$1.45 \cdot 10^{-4}$	5.09
64	$1.36 \cdot 10^{-4}$	2.79	$1.16 \cdot 10^{-5}$	5.48	$2.78 \cdot 10^{-5}$	5.21
128	$4.83 \cdot 10^{-5}$	2.81	$2.11 \cdot 10^{-6}$	5.47	$5.24 \cdot 10^{-6}$	5.31
256	$1.71 \cdot 10^{-5}$	2.82	$3.83 \cdot 10^{-7}$	5.51	$9.72 \cdot 10^{-7}$	5.39

Example 5.1. We first consider the following boundary value problem:

$$(D_*^{0.5}y)(t) + a_0(t)y(t) = f(t), \qquad y(0) + y(1) = 2, \quad 0 \le t \le 2, \tag{5.1}$$

where

$$a_0(t) := t^{0.5}, \qquad f(t) := \frac{2}{\Gamma(1.5)}\, t^{0.5} + 2\, t^{1.5}.$$

This is a special problem of (1.1), (1.2) with $n = p = 1,\ \alpha_1 = 0.5,\ \alpha_0 = 0,\ b = 2,\ b_1 = 1,\ n_0 = n_1 = 0,\ \alpha_{00} = \beta_{00} = 1$ and $\gamma_0 = 2$. Clearly, $a_0, f \in C^{q,\nu}(0, 2]$ with $\nu = 0.5$ and arbitrary $q \in \mathbb{N}$.

To solve (5.1) by (3.4), (3.6) we set $z := D_*^{0.5}y$. Then it follows from (2.6)–(2.8) that $y(t) = (Gz)(t) + 1$, where $(Gz)(t) = (J^{0.5}z)(t) - 0.5\,(J^{0.5}z)(1),\ 0 \le t \le 2$. For z we have Eq. (2.9) with $Tz = -a_0 Gz$ and $g = f - a_0$. Approximations $z_N \in S_{m-1}^{(-1)}(\Pi_N)$ for $m = 2$ and $N \in \mathbb{N}$ to the solution z of Eq. (2.9) on the interval $[0, 2]$ are found by (3.5) using $m = 2$ and (3.2) with $\eta_1 = (3 - \sqrt{3})/6,\ \eta_2 = 1 - \eta_1$, the knots of the Gaussian quadrature formula (4.1). Actually, $z_N(t_{jk}) = \hat{z}_N(t_{jk}) = c_{jk}$ $(k = 1, 2,\ j = 1, \ldots, N)$ and $z_N(t)$ for $t \in [0,\ 2]$ are determined by (3.8) and (3.9), respectively. After that the approximate solution y_N for the problem (5.1) has been found by the formula $y_N(t) = (Gz_N)(t) + 1,\ 0 \le t \le 2$.

In Table 5.1 some results of numerical experiments for different values of the parameters N and r are presented. The errors ε_N and $\hat{\varepsilon}_N$ in Table 5.1 are calculated as follows:

$$\begin{aligned} \varepsilon_N &:= \max_{j=1,\ldots,N}\ \max_{k=0,\ldots,10} |y(\tau_{jk}) - y_N(\tau_{jk})|, \\ \hat{\varepsilon}_N &:= \max_{j=1,\ldots,N}\ \max_{k=1,2} |z(t_{jk}) - \hat{z}_N(t_{jk})| = \max_{j=1,\ldots,N}\ \max_{k=1,2} |z(t_{jk}) - z_N(t_{jk})|, \end{aligned} \tag{5.2}$$

where $\tau_{jk} := t_{j-1} + k(t_j - t_{j-1})/10,\ k = 0, \ldots, 10,\ j = 1, \ldots, N$ (the grid points t_j and collocation points t_{jk} are determined by (3.1) and (3.2), respectively). In (5.2) we have taken into account that the exact solution of (5.1) is $y = 2\,t$ and thus $z = D_*^{0.5}y = (2/\Gamma(1.5))\, t^{0.5}$. The ratios

$$\varrho_N := \frac{\varepsilon_{N/2}}{\varepsilon_N}, \qquad \hat{\varrho}_N := \frac{\hat{\varepsilon}_{N/2}}{\hat{\varepsilon}_N}, \tag{5.3}$$

characterizing the observed convergence rate, are also presented.

Since $\alpha_p = \alpha_1 = 0.5,\ \beta = \max\{\alpha_{p-1}, n_1\} = 0$ and $\nu = 0.5$ we obtain from (4.10) and (4.14) that, for sufficiently large N,

$$\max\{\varepsilon_N, \hat{\varepsilon}_N\} \le c \begin{cases} N^{-r} & \text{if } 1 \le r < 2.5, \\ N^{-2.5} & \text{if } r \ge 2.5. \end{cases} \tag{5.4}$$

Due to (5.4) the ratios ϱ_N and $\hat{\varrho}_N$ for $r = 1,\ r = 2$ and $r = 2.5$ ought to be approximatively 2, 4 and $2^{2,5} \approx 5, 66$, respectively. As we can see from Table 5.1 the estimate (4.10) expresses well enough the actual rate of convergence of y_N to y (only the decrease of $\hat{\varepsilon}_N$ to 0 is for $r = 1$ and $r = 2$ somewhat faster than we would expect by the estimate (4.14)).

Table 5.2
Results for the problem (5.5).

N	$r=1$		$r=2$		$r=2.5$	
	ε_N	ϱ_N	ε_N	ϱ_N	ε_N	ϱ_N
4	$3.64\cdot10^{-4}$	5.07	$1.57\cdot10^{-4}$	11.82	$2.99\cdot10^{-4}$	9.52
8	$6.99\cdot10^{-5}$	5.20	$1.31\cdot10^{-5}$	11.95	$2.29\cdot10^{-5}$	13.08
16	$2.12\cdot10^{-5}$	3.29	$1.35\cdot10^{-6}$	8.70	$1.80\cdot10^{-6}$	12.70
32	$7.25\cdot10^{-6}$	2.93	$1.65\cdot10^{-7}$	8.18	$1.86\cdot10^{-7}$	9.66
64	$2.52\cdot10^{-6}$	2.87	$2.28\cdot10^{-8}$	7.25	$2.46\cdot10^{-8}$	7.57
128	$8.89\cdot10^{-7}$	2.84	$3.41\cdot10^{-9}$	6.69	$3.79\cdot10^{-9}$	6.54
256	$3.14\cdot10^{-7}$	2.83	$5.36\cdot10^{-10}$	6.35	$6.33\cdot10^{-10}$	5.99
N	$\hat{\varepsilon}_N$	$\hat{\varrho}_N$	$\hat{\varepsilon}_N$	$\hat{\varrho}_N$	$\hat{\varepsilon}_N$	$\hat{\varrho}_N$
4	$1.74\cdot10^{-2}$	1.85	$4.91\cdot10^{-3}$	3.56	$3.98\cdot10^{-3}$	4.64
8	$9.32\cdot10^{-3}$	1.87	$1.31\cdot10^{-3}$	3.74	$7.94\cdot10^{-4}$	5.01
16	$4.91\cdot10^{-3}$	1.90	$3.40\cdot10^{-4}$	3.86	$1.49\cdot10^{-4}$	5.34
32	$2.55\cdot10^{-3}$	1.92	$8.65\cdot10^{-5}$	3.93	$2.69\cdot10^{-5}$	5.52
64	$1.31\cdot10^{-3}$	1.94	$2.18\cdot10^{-5}$	3.96	$4.81\cdot10^{-6}$	5.60
128	$6.70\cdot10^{-4}$	1.96	$5.48\cdot10^{-6}$	3.98	$8.54\cdot10^{-7}$	5.63
256	$3.40\cdot10^{-4}$	1.97	$1.37\cdot10^{-6}$	3.99	$1.51\cdot10^{-7}$	5.65

Example 5.2. Secondly we consider the following boundary value problem for the Bagley–Torvik equation:

$$y''(t) + (D_*^{1.5}y)(t) + y(t) = f(t), \qquad y(0) = 1, \qquad y(1) = 2, \quad 0 \le t \le 1, \tag{5.5}$$

where

$$f(t) := \frac{15}{4}\,t^{0.5} + \frac{15}{8}\sqrt{\pi}t + t^{2.5} + 1.$$

Let $z := y''$. Then a solution of the problem (5.5) is given by the formula $y(t) = (Gz)(t) + 1 + t$ where $(Gz)(t) = (J^2z)(t) - t(J^2z)(1)$ and z satisfies Eq. (2.9) in which $Tz = -Gz - J^{0.5}z$ and $g(t) = f(t) - 1 - t$. The values $z_N(t_{jk}) = c_{jk}$ $(k = 1, 2,\ j = 1, \dots, N,\ N \in \mathbb{N})$ are calculated by (3.9) using $m = 2,\ \eta_1 = (3 - \sqrt{3})/6$, $\eta_2 = 1 - \eta_1$ and $b = b_1 = 1$. After that the approximation y_N to the solution y of (5.5) has been found by the formula $y_N(t) = (Gz_N)(t) + 1 + t$.

In Table 5.2 the errors $\varepsilon_N,\ \hat{\varepsilon}_N$ and the ratios ϱ_N, $\hat{\varrho}_N$ determined by (5.2) and (5.3) are presented. Also in this case we have used the exact solution $y = t^{2.5} + 1$ of the problem (5.5) and its second derivative $z = y'' = 3.75t^{0.5}$. Since in this example $\alpha_2 = 2,\ \beta = 1.5$ and $\nu = 0.5$ we get from (4.10) and (4.14) the estimate (5.4), too. Thus, we would expect that our method for $r = 1,\ r = 2$ and $r = 2.5$ would have order 2, 4 and $2^{2.5} \approx 5.66$, respectively. These expectations are confirmed by the results in Table 5.2. Actually, we see from Table 5.2, that for this example the estimate (4.14) is in good accordance with the actual rate of the convergence of $\hat{z}_N$ to z, but the convergence of y_N to y is for $r = 1$ and $r = 2$ faster than it is predicted by the estimate (4.10).

References

[1] K. Diethelm, The Analysis of Fractional Differential Equations, in: Lecture Notes in Mathematics, vol. 2004, Springer, Berlin, 2010.
[2] H. Brunner, A. Pedas, G. Vainikko, Piecewise polynomial collocation methods for linear Volterra integro-differential equations with weakly singular kernels, SIAM J. Numer. Anal. 39 (2001) 957–982.
[3] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, in: North-Holland Mathematics Studies, vol. 204, Elsevier, Amsterdam, 2006.
[4] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.
[5] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives. Theory and Applications, Gordon and Breach Science Publishers, Yverdon, 1993.
[6] R. Agarwal, M. Benchohra, H. Hamani, A survey on existence results for boundary value problems of nonlinear fractional equations and inclusion, Acta Appl. Math. 109 (2010) 973–1033.
[7] R.P. Agarwal, B. Ahmad, Existence theory for anti-periodic boundary value problems of fractional differential equations and inclusions, Comput. Math. Appl. 62 (2011) 1200–1214.
[8] B. Ahmad, J.J. Nieto, J. Pimentel, Some boundary value problems of fractional differential equations and inclusions, Comput. Math. Appl. 62 (2011) 1238–1250.
[9] M. ur Rehman, R.A. Khan, N.A. Asif, Three point boundary value problems for nonlinear fractional differential equations, Acta Math. Sci. 31B (2011) 1337–1346.
[10] X. Yang, Z. Wei, W. Dong, Existence of positive solutions for the boundary value problem of nonlinear fractional differential equations, Commun. Nonlinear Sci. Numer. Simul. 17 (2012) 85–92.
[11] J.T. Edwards, N.J. Ford, A.C. Simpson, The numerical solution of linear multi-term fractional differential equations: systems of equations, J. Comput. Appl. Math. 148 (2002) 401–418.
[12] K. Diethelm, N.J. Ford, Numerical solution of the Bagley–Torvik equation, BIT 42 (2002) 490–507.
[13] K. Diethelm, N.J. Ford, A.D. Freed, Detailed error analysis for a fractional Adams method, Numer. Algorithms 36 (2004) 31–52.
[14] N.J. Ford, J.A. Connoly, Systems-based decomposition schemes for approximate solution of multi-term fractional differential equations, J. Comput. Appl. Math. 229 (2009) 382–391.
[15] R. Garrappa, On some explicit Adams multistep methods for fractional differential equations, J. Comput. Appl. Math. 229 (2009) 392–399.

[16] A. Pedas, E. Tamme, Spline collocation methods for linear multi-term fractional differential equations, J. Comput. Appl. Math. 236 (2011) 167–176.
[17] N.J. Ford, M.L. Morgado, Fractional boundary value problems: analysis and numerical methods, Fract. Calc. Appl. Anal 14 (2011) 554–567.
[18] E.H. Doha, A.H. Bhrawy, S.S. Ezz-Eldien, A Chebyshev spectral method based on operational matrix for initial and boundary value problems of fractional order, Comput. Math. Appl. 62 (2011) 2364–2373.
[19] G.M. Al-Mdallal, M.I. Syam, M.N. Anwar, A collocation-shooting method for solving fractional boundary value problems, Commun. Nonlinear Sci. Numer. Simul. 15 (2010) 3814–3822.
[20] M. ur Rehman, R.A. Khan, A Numerical method for solving boundary value problems for fractional differential equations, Appl. Math. Model. 36 (2012) 894–907.
[21] G. Vainikko, Multidimensional Weakly Singular Integral Equations, in: Lecture Notes in Mathematics, vol. 1549, Springer, Berlin, 1993.
[22] H. Brunner, Collocation Methods for Volterra Integral and Related Functional Equations, in: Cambridge Monographs on Applied and Computational Mathematics, vol. 15, Cambridge University Press, Cambridge, UK, 2004.
[23] M. Kolk, A. Pedas, G. Vainikko, High-order methods for Volterra integral equations with general weak singularities, Numer. Funct. Anal. Optim. 30 (2009) 1002–1024.
[24] A. Pedas, E. Tamme, Spline collocation method for integro-differential equations with weakly singular kernels, J. Comput. Appl. Math. 197 (2006) 253–269.
[25] A. Pedas, E. Tamme, Product integration for weakly singular integro-differential equations, Math. Model. Anal. 16 (2011) 153–172.
[26] A. Pedas, E. Tamme, On the convergence of spline collocation methods for solving fractional differential equations, J. Comput. Appl. Math. 235 (2011) 3502–3514.